ENRICHED STONE-TYPE DUALITIES

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In memory of Klaus Keimel

Abstract. A common feature of many duality results is that the involved equivalence functors are liftings of hom-functors into the two-element space resp. lattice. Due to this fact, we can only expect dualities for categories cogenerated by the two-element set with an appropriate structure. A prime example of such a situation is Stone's duality theorem for Boolean algebras and Boolean spaces, the latter being precisely those compact Hausdorff spaces which are cogenerated by the two-element discrete space. In this paper we aim for a systematic way of extending this duality theorem to categories including all compact Hausdorff spaces. To achieve this goal, we combine duality theory and quantale-enriched category theory. Our main idea is that, when passing from the two-element discrete space to a cogenerator of the category of compact Hausdorff spaces, all other involved structures should be substituted by corresponding enriched versions. Accordingly, we work with the unit interval $[0,1]$ and present duality theory for ordered and metric compact Hausdorff spaces and (suitably defined) finitely cocomplete categories enriched in $[0,1]$.

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1. Introduction

In [Baez and Dolan, 2001], the authors make the seemingly paradoxical observation that "...an equation is only interesting or useful to the extent that the two sides are different!". Undoubtedly, a moment's thought convinces us that an equation like $e^{i\omega} = \cos(\omega) + i\sin(\omega)$ is far more interesting than the rather dull statement that $3 = 3$. A comparable remark applies if we go up in dimension: equivalent categories are thought to be essentially equal, but an equivalence is of greater interest if the involved categories look different. Numerous examples of equivalences of "different" categories relate a category X and the

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dual of a category A. Such an equivalence is called a ***dual equivalence*** or simply a ***duality***, and is usually denoted by $\mathsf{X} \simeq \mathsf{A}^{\mathrm{op}}$. Like for every other equivalence, a duality allows us to transport properties from one side to the other. The presence of the dual category on one side is often useful since our knowledge of properties of a category is typically asymmetric. Indeed, many "everyday categories" admit a representable and hence limit preserving functor to Set. Therefore in these categories limits are "easy"; however, colimits are often "hard". In these circumstances, an equivalence $\mathsf{X} \simeq \mathsf{A}^{\mathrm{op}}$ together with the knowledge of limits in A help us understand colimits in X. The dual situation, where colimits are "easy" and limits are "hard", frequently emerges in the context of coalgebras. For example, the category $\mathrm{CoAlg}(V)$ of coalgebras for the Vietoris functor V on the category BooSp of Boolean spaces and continuous functions is known to be equivalent to the dual of the category BAO with objects Boolean algebras B with an operator $h : B \to B$ satisfying the equations

$$h(\bot) = \bot \qquad \text{and} \qquad h(x \vee y) = h(x) \vee h(y),$$

and with morphisms the Boolean homomorhisms which also preserve the additional unary operation (see [Halmos, 1956]). It is a trivial observation that BAO is a category of algebras over Set defined by a (finite) set of operations and a collection of equations; every such category is known to be complete and cocomplete. Notably, the equivalence $\mathrm{CoAlg}(V) \simeq \mathsf{BAO}^{\mathrm{op}}$ allows to conclude the non-trivial fact that $\mathrm{CoAlg}(V)$ is complete. This argument also shows that, starting with a category X, the category A in a dual equivalence $\mathsf{X} \simeq \mathsf{A}^{\mathrm{op}}$ does not need to be a familiar category. It is certainly beneficial that $\mathsf{A} = \mathsf{BAO}$ is a well-known category; however, every algebraic category describable by a set of operations would be sufficient to conclude completeness of $\mathsf{X} = \mathrm{CoAlg}(V)$. We refer to [Kupke *et al.*, 2004; Bonsangue *et al.*, 2007] for more examples of dualities involving categories of coalgebras.

The example above as well as the classical Stone-dualities for Boolean algebras and distributive lattices (see [Stone, 1936, 1938a,b]) are obtained using the two-element space or the two-element lattice. Due to this fact, we can only expect dualities for categories cogenerated by $2 = \{0, 1\}$ with an appropriate structure. For instance, the category BooSp is the full subcategory of the category $\mathsf{CompHaus}$ of compact Hausdorff spaces and continuous maps defined by those spaces X where the cone $(f : X \to 2)_f$ is point-separating and initial. In order to obtain duality results involving all compact Hausdorff spaces, we need to work with a cogenerator of $\mathsf{CompHaus}$ rather than the 2-element discrete space. Of course, this is exactly the perspective taken in the classical Gelfand duality theorem (see [Gelfand, 1941]) or in several papers on lattices of continuous functions (see [Kaplansky, 1947, 1948] and [Banaschewski, 1983]) that consider functions into the unit disc or the unit interval. However, in these approaches, the objects of the dual category of $\mathsf{CompHaus}$ do not appear immediately as generalisations of Boolean algebras.

This is the right moment to mention another cornerstone of our work: the theory of quantale-enriched categories. Our main motivation stems from Lawvere's seminal paper [Lawvere, 1973] that investigates metric spaces as categories enriched in the quantale $[0, \infty]$. Keeping in mind that ordered sets[1] are categories enriched in the two-element quantale 2, our thesis is *that the passage from the two-element space to the compact Hausdorff space* $[0, \infty]$ *should be matched by a move from ordered structures to metric structures on the other side.* In fact, we claim that some results about lattices of real-valued continuous functions hiddenly talk about (ultra)metric spaces; for instance, in Section 2, we point out how to interpret Propositions 2 and 3 of [Banaschewski, 1983] in this way. Roughly speaking, in analogy with the results for the two-element space, we are looking for an equivalence functor (or at least a full embedding)

$$\mathsf{CompHaus} \longrightarrow (\text{metric spaces with some (co)completeness properties})^{\mathrm{op}}$$

and, more generally, with $\mathsf{Ord_sComp}$ denoting the category of separated (=anti-symmetric) ordered compact Hausdorff spaces and monotone continuous maps, a full embedding

$$\mathsf{Ord_sComp} \longrightarrow (\text{metric spaces with some (co)completeness properties})^{\mathrm{op}}.$$

[1] In this paper, an order relation need not be anti-symmetric; we require only reflexivity and transitivity.

Inspired by [Halmos, 1956], we obtain this as a restriction of a more general result about a full embedding of the Kleisli category [Kleisli, 1965] $\mathsf{Ord_sComp}_{\mathbb{V}}$ of the Vietoris monad $\mathbb{V}$ on $\mathsf{Ord_sComp}$:

$$\mathsf{Ord_sComp}_{\mathbb{V}} \longrightarrow (\text{“finitely cocomplete” metric spaces})^{\mathrm{op}}.$$

The Vietoris monad has its roots in [Vietoris, 1922] and various generalisations of this “power construction” are extensively studied in [Schalk, 1993]. The notion of “finitely cocomplete metric space” should be considered as the metric counterpart to semi-lattice, and “metric space with some (co)completeness properties” as the metric counterpart to (distributive) lattice. This way we also exhibit the algebraic nature of the dual category of $\mathsf{Ord_sComp}$ which resembles the classical result stating that $\mathsf{CompHaus}^{\mathrm{op}}$ is a $\aleph_1$-ary variety (see [Isbell, 1982; Marra and Reggio, 2017]).

For technical reasons, we consider structures enriched in a quantale based on $[0,1]$ rather than in $[0,\infty]$; nevertheless, since the lattices $[0,1]$ and $[0,\infty]$ are isomorphic, we still talk about metric spaces. In Section 2 we recall the principal facts about quantale-enriched categories needed in this paper, and in Section 3 we present the classification of continuous quantale structures on the unit interval $[0,1]$ obtained in [Faucett, 1955] and [Mostert and Shields, 1957]. Since the Vietoris monad $\mathbb{V}$ on the category $\mathsf{Ord_sComp}$ plays a key role in the results of Section 6, we provide the necessary background material in Section 4. We review duality theory in Section 5; in particular, for a monad $\mathbb{T}$, we explain the connection between functors $\mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$ and certain $\mathbb{T}$-algebras. After these introductory parts, in Section 6 we develop a duality theory for the Kleisli category $\mathsf{Ord_sComp}_{\mathbb{V}}$ of $\mathbb{V}$. We found a first valuable hint for doing so in [Shapiro, 1992] where the author gives a functional representation of the classical Vietoris monad on $\mathsf{CompHaus}$ using the algebraic structure on the non-negative reals. Inspired by this result, for every continuous quantale structure on $[0,1]$, we obtain a functional representation of the Vietoris monad on $\mathsf{Ord_sComp}$, which leads to a full embedding of $\mathsf{Ord_sComp}_{\mathbb{V}}$ into a category of monoids of finitely cocomplete $[0,1]$-categories. We also identify the continuous functions in $\mathsf{Ord_sComp}_{\mathbb{V}}$ as precisely the monoid homomorphisms on the other side. Section 7 presents a Stone–Weierstraß type theorem for $[0,1]$-categories which helps us to establish a dual equivalence involving the category $\mathsf{Ord_sComp}_{\mathbb{V}}$. Finally, since we moved from order structures to structures enriched in $[0,1]$, it is only logical to also substitute the Vietoris monad, which is based on functions $X \to 2$, by a monad that uses functions of type $X \to [0,1]$ defined on metric generalisations of ordered compact Hausdorff spaces. In Sections 8 and 9 we extend our setting from ordered compact Hausdorff spaces to “metric compact Hausdorff spaces” and consider the enriched Vietoris monads introduced in [Hofmann, 2014]. Denoting these monads by $\mathbb{V}$ as well, in analogy to the ordered case, for certain quantale structures on $[0,1]$ we obtain a full embedding

$$(\text{metric compact Hausdorff spaces})_{\mathbb{V}} \longrightarrow (\text{“finitely cocomplete” } [0,1]\text{-categories})^{\mathrm{op}}.$$

Last but not least, we would like to point out that this is not the first work transporting classical duality results into the realm of metric spaces. An approach version (see [Lowen, 1997]) of the duality between the categories of sober spaces and continuous maps and of spatial frames and homomorphisms is obtained in [Banaschewski *et al.*, 2006] and extensively studied in [Van Olmen, 2005] (see also [Van Olmen and Verwulgen, 2010]). By definition, an approach frame is a frame with some actions of $[0,\infty]$; keeping in mind the results of Section 2, we can describe approach frames as particular (co)complete metric spaces. This point of view is taken in [Hofmann and Stubbe, 2011].

2. Enriched categories as actions

To explain the passage from the ordered to the metric case, it is convenient to view ordered sets and metric spaces as instances of the same notion, namely that of a quantale-enriched category. All material presented here is well-known, we refer to the classical sources [Eilenberg and Kelly, 1966], [Lawvere, 1973] and [Kelly, 1982]. A very extensive presentation of this theory in the quantaloid-enriched case can be found in [Stubbe, 2005, 2006, 2007]. We would also like to point the reader to [Kelly and Lack, 2000], [Kelly and Schmitt, 2005] and [Clementino and Hofmann, 2009] where enriched categories with certain colimits are studied.

Definition 2.1. A (commutative and unital) ***quantale*** $\mathcal{V}$ is a complete lattice which carries the structure of a commutative monoid $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ with unit element $k \in \mathcal{V}$ such that $u \otimes - : \mathcal{V} \to \mathcal{V}$ preserves suprema, for each $u \in \mathcal{V}$.

Hence, every monotone map $u \otimes - : \mathcal{V} \to \mathcal{V}$ has a right adjoint $\hom(u, -) : \mathcal{V} \to \mathcal{V}$ which is characterised by

$$u \otimes v \leq w \iff v \leq \hom(u, w),$$

for all $v, w \in \mathcal{V}$.

Definition 2.2. Let $\mathcal{V}$ be a quantale. A $\mathcal{V}$***-category*** is a pair (X, a) consisting of a set X and a map $a : X \times X \to \mathcal{V}$ satisfying

$$k \leq a(x, x) \qquad \text{and} \qquad a(x, y) \otimes a(y, z) \leq a(x, z),$$

for all $x, y, z \in X$. Given $\mathcal{V}$-categories (X, a) and (Y, b), a $\mathcal{V}$***-functor*** $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that

$$a(x, y) \leq b(f(x), f(y)),$$

for all $x, y \in X$.

For every $\mathcal{V}$-category (X, a), $a^\circ(x, y) = a(y, x)$ defines another $\mathcal{V}$-category structure on X, and the $\mathcal{V}$-category $(X, a)^{\mathrm{op}} := (X, a^\circ)$ is called the ***dual*** of (X, a). Clearly, $\mathcal{V}$-categories and $\mathcal{V}$-functors define a category, denoted as $\mathcal{V}\text{-}\mathsf{Cat}$. The category $\mathcal{V}\text{-}\mathsf{Cat}$ is complete and cocomplete, and the canonical forgetful functor $\mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Set}$ preserves limits and colimits. The quantale $\mathcal{V}$ becomes a $\mathcal{V}$-category with structure $\hom : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$. For every set S, we can form the S-power $\mathcal{V}^S$ of $\mathcal{V}$ which has as underlying set all functions $h : S \to \mathcal{V}$, and the $\mathcal{V}$-category structure $[-, -]$ is given by

$$[h, l] = \bigwedge_{s \in S} \hom(h(s), l(s)),$$

for all $h, k : S \to \mathcal{V}$.

Examples 2.3. Our principal examples are the following.

(1) The two-element Boolean algebra $2 = \{0, 1\}$ of truth values with $\otimes$ given by "and" &. Then $\hom(u, v) = (u \implies v)$ is implication. The category $2\text{-}\mathsf{Cat}$ is equivalent to the category Ord of ordered sets and monotone maps.

(2) The complete lattice $[0, \infty]$ ordered by the "greater or equal" relation $\geqslant$ (so that the infimum of two numbers is their maximum and the supremum of $S \subseteq [0, \infty]$ is given by $\inf S$), with multiplication $\otimes = +$. In this case we have

$$\hom(u, v) = v \ominus u := \max(v - u, 0).$$

For this quantale, a $[0, \infty]$-category is a generalised metric space à la Lawvere and a $[0, \infty]$-functor is a non-expansive map (see [Lawvere, 1973]). We denote this category by Met.

(3) Of particular interest to us is the complete lattice $[0, 1]$ with the usual "less or equal" relation $\leq$, which is isomorphic to $[0, \infty]$ via the map $[0, 1] \to [0, \infty]$, $u \mapsto -\ln(u)$ where $-\ln(0) = \infty$. As the examples below show, metric, ultrametric and bounded metric spaces appear as categories enriched in a quantale based on this lattice. More in detail, we consider the following quantale operations on $[0, 1]$ with neutral element 1.

(a) The tensor $\otimes = *$ is the multiplication and then

$$\hom(u, v) = v \oslash u := \begin{cases} \min(\frac{v}{u}, 1) & \text{if } u \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Via the isomorphism $[0,1] \simeq [0,\infty]$, this quantale is isomorphic to the quantale $[0,\infty]$ described above, hence $[0,1]\text{-}\mathsf{Cat} \simeq \mathsf{Met}$.

(b) The tensor $\otimes = \wedge$ is infimum and then

$$\hom(u,v) = \begin{cases} 1 & \text{if } u \leq v, \\ v & \text{otherwise.} \end{cases}$$

In this case, the isomorphism $[0,1] \simeq [0,\infty]$ establishes an equivalence between $[0,1]\text{-}\mathsf{Cat}$ and the category UMet of ultrametric spaces and non-expansive maps.

(c) The tensor $\otimes = \odot$ is the ***Łukasiewicz tensor*** given by $u \odot v = \max(0, u+v-1)$, here $\hom(u,v) = \min(1, 1-u+v) = 1-\max(0,u-v)$. Via the lattice isomorphism $[0,1] \to [0,1]$, $u \mapsto 1-u$, this quantale is isomorphic to the quantale $[0,1]$ with "greater or equal" relation $\geqslant$ and tensor $u \otimes v = \min(1, u+v)$ truncated addition. This observation identifies $[0,1]\text{-}\mathsf{Cat}$ as the category BMet of bounded-by-1 metric spaces and non-expansive maps.

Every $\mathcal{V}$-category (X,a) carries a natural order defined by

$$x \leq y \text{ whenever } k \leq a(x,y),$$

which can be extended pointwise to $\mathcal{V}$-functors making $\mathcal{V}\text{-}\mathsf{Cat}$ a *2-category*. Therefore we can talk about adjoint $\mathcal{V}$-functors; as usual, $f : (X,a) \to (Y,b)$ is left adjoint to $g : (Y,b) \to (X,a)$, written as $f \dashv g$, whenever $1_X \leq gf$ and $fg \leq 1_Y$. Equivalently, $f \dashv g$ if and only if

$$b(f(x),y) = a(x,g(y)),$$

for all $x \in X$ and $y \in Y$. We note that maps f and g between $\mathcal{V}$-categories satisfying the equation above are automatically $\mathcal{V}$-functors.

The natural order of $\mathcal{V}$-categories defines a faithful functor $\mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Ord}$. A $\mathcal{V}$-category is called ***separated*** whenever its underlying ordered set is anti-symmetric, and we denote by $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{s}}$ the full subcategory of $\mathcal{V}\text{-}\mathsf{Cat}$ defined by all separated $\mathcal{V}$-categories. Tautologically, an ordered set is separated if and only if it is anti-symmetric. Hence, $\mathsf{Ord}_{\mathrm{s}}$ denotes the category of all separated ordered sets and monotone maps. In the sequel we will frequently consider separated $\mathcal{V}$-categories in order to guarantee that adjoints are unique. We note that the underlying order of the $\mathcal{V}$-category $\mathcal{V}$ is just the order of the quantale $\mathcal{V}$, and the order of $\mathcal{V}^S$ is calculated pointwise. In particular, $\mathcal{V}^S$ is separated.

Definition 2.4. A $\mathcal{V}$-category (X,a) is called ***$\mathcal{V}$-copowered*** whenever the $\mathcal{V}$-functor $a(x,-) : (X,a) \to (\mathcal{V}, \hom)$ has a left adjoint $x \otimes - : (\mathcal{V},\hom) \to (X,a)$ in $\mathcal{V}\text{-}\mathsf{Cat}$, for every $x \in X$.

We note that this operation is better known under the name "$\mathcal{V}$-tensored"; however, we prefer to use the designation "$\mathcal{V}$-copowered" since it is a special case of a *co*limit. Elementwise, this means that, for all $x \in X$ and $u \in \mathcal{V}$, there exists some element $x \otimes u \in X$, called the u-copower of x, such that

$$a(x \otimes u, y) = \hom(u, a(x,y)),$$

for all $y \in X$.

Example 2.5. The $\mathcal{V}$-category $\mathcal{V}$ is $\mathcal{V}$-copowered, with copowers given by the multiplication of the quantale $\mathcal{V}$. More generally, for every set S, the $\mathcal{V}$-category $\mathcal{V}^S$ is $\mathcal{V}$-copowered: for every $h \in \mathcal{V}^S$ and $u \in \mathcal{V}$, the u-copower of h is given by $(h \otimes u)(x) = h(x) \otimes u$, for all $x \in S$.

Remark 2.6. If (X,a) is a $\mathcal{V}$-copowered $\mathcal{V}$-category, then, for every $x \in X$ and $u = \bot$ the bottom element of $\mathcal{V}$, we have

$$a(x \otimes \bot, y) = \hom(\bot, a(x,y)) = \top$$

for all $y \in X$. In particular, $x \otimes \bot$ is a bottom element of the $\mathcal{V}$-category (X,a).

Every $\mathcal{V}$-copowered and separated $\mathcal{V}$-category comes equipped with an action $\otimes : X \times \mathcal{V} \to X$ of the quantale $\mathcal{V}$ satisfying

(1) $x \otimes k = x$,
(2) $(x \otimes u) \otimes v = x \otimes (u \otimes v)$,
(3) $x \otimes \bigvee_{i \in I} u_i = \bigvee_{i \in I} (x \otimes u_i)$;

for all $x \in X$ and $u, v, u_i \in \mathcal{V}$ $(i \in I)$. Conversely, given a separated ordered set X with an action $\otimes : X \times \mathcal{V} \to X$ satisfying the three conditions above, one defines a map $a : X \times X \to \mathcal{V}$ by $x \otimes - \dashv a(x, -)$, for all $x \in X$. It is easy to see that (X, a) is a $\mathcal{V}$-copowered $\mathcal{V}$-category whose order is the order of X and where copowers are given by the action of X. Writing $\mathcal{V}\text{-}\mathsf{CoPow}_{\mathrm{s}}$ for the category of $\mathcal{V}$-copowered and separated $\mathcal{V}$-categories and copower-preserving $\mathcal{V}$-functors and $\mathsf{Ord}_{\mathrm{s}}^{\mathcal{V}}$ for the category of separated ordered sets X with an action $\otimes : X \times \mathcal{V} \to X$ satisfying the three conditions above and action-preserving monotone maps, the above construction yields an isomorphism

$$\mathcal{V}\text{-}\mathsf{CoPow}_{\mathrm{s}} \simeq \mathsf{Ord}_{\mathrm{s}}^{\mathcal{V}}.$$

We also note that the inclusion functor $\mathcal{V}\text{-}\mathsf{CoPow}_{\mathrm{s}} \to \mathcal{V}\text{-}\mathsf{Cat}$ is monadic.

Remark 2.7. The identification of certain metric spaces as ordered sets with an action of $[0, 1]$ allows us to spot the appearance of metric spaces where it does not seem obvious at first sight. For instance, [Banaschewski, 1983] considers the distributive lattice DX of continuous functions from a compact Hausdorff space X into the unit interval $[0, 1]$, and, for a continuous map $f : X \to Y$, the lattice homomorphism $Df : DY \to DX$, $\psi \mapsto \psi \cdot f$ is given by composition with f. In [Banaschewski, 1983, Proposition 2] it is shown that a lattice homomorphism $\varphi : DY \to DX$ is of the form $\varphi = Df$, for some continuous map $f : X \to Y$, if and only if φ preserves constant functions. Subsequently, [Banaschewski, 1983] consideres the algebraic theory of distributive lattices augmented by constants, one for each element of $[0, 1]$; and eventually obtains a duality result for compact Hausdorff spaces. Motivated by the considerations in this section, instead of adding constants we will consider DX as a lattice equipped with the action of $[0, 1]$ defined by

$$(f \otimes u)(x) = f(x) \wedge u,$$

and then [Banaschewski, 1983, Proposition 2] tells us that the lattice homomorphisms $\varphi : DY \to DX$ of the form $\varphi = Df$ are precisely the action-preserving ones. Hence, Banaschewski's result can be reinterpreted in terms of $[0, 1]$-copowered ultrametric spaces.

The notion of copower in a $\mathcal{V}$-category (X, a) is a special case of a weighted colimit in (X, a), as we recall next. In the remainder of this section we write G to denote the $\mathcal{V}$-category $(1, k)$; note that G is a generator in $\mathcal{V}\text{-}\mathsf{Cat}$.

For a quantale $\mathcal{V}$ and sets X, Y, a ***$\mathcal{V}$-relation*** from X to Y is a map $X \times Y \to \mathcal{V}$ and it will be represented by $X \relbar\joinrel\mapsto Y$. As for ordinary relations, $\mathcal{V}$-relations can be composed via "matrix multiplication". That is, for $r : X \relbar\joinrel\mapsto Y$ and $s : Y \relbar\joinrel\mapsto Z$, the composite $s \cdot r : X \relbar\joinrel\mapsto Z$ is calculated pointwise by

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z),$$

for every $x \in X$ and $z \in Z$. We note that the structure of a $\mathcal{V}$-category is by definition a reflexive and transitive $\mathcal{V}$-relation, since the axioms dictate that, for a $\mathcal{V}$-category (X, a), $1_X \le a$ and $a \cdot a \le a$. A $\mathcal{V}$-relation $r : X \relbar\joinrel\mapsto Y$ between $\mathcal{V}$-categories (X, a) and (Y, b) is called a ***$\mathcal{V}$-distributor*** (called bimodule in [Lawvere, 1973]) if $r \cdot a \le r$ and $b \cdot r \le r$, and we write $r : (X, a) \circ\!\!\rightarrow (Y, b)$.

A ***weighted colimit diagram*** in X is given by a $\mathcal{V}$-category A together with a $\mathcal{V}$-functor $h : A \to X$ and a $\mathcal{V}$-distributor $\psi : A \circ\!\!\rightarrow G$, the latter is called the ***weight*** of the diagram. A ***colimit*** of such a diagram is an element $x_0 \in X$ such that, for all $x \in X$,

$$a(x_0, x) = \bigwedge_{z \in A} \hom(\psi(z), a(h(z), x)).$$

If a weighted colimit diagram has a colimit, then this colimit is unique up to equivalence. A $\mathcal{V}$-functor $f : X \to Y$ between $\mathcal{V}$-categories ***preserves*** the colimit of this diagram whenever $f(x_0)$ is a colimit of the weighted colimit diagram in Y given by $fh : A \to Y$ and $\psi : A \mathrel{\circ\!\!\to} G$. For more details we refer to [Kelly, 1982].

Examples 2.8. (1) For $A = G$, a weighted colimit diagram in X is given by an element $x : G \to X$ and an element $u : G \mathrel{\circ\!\!\to} G$ in $\mathcal{V}$, a colimit of this diagram is the u-copower $x \otimes u$ of x.

(2) For a family $h : I \to X$, $i \mapsto x_i$ in X we consider the distributor $\psi : I \mathrel{\circ\!\!\to} G$ defined by $\psi(z) = k$, for all $z \in I$. Then $\bar{x}$ is a colimit of this diagram precisely when

$$a(\bar{x}, x) = \bigwedge_{i \in I} a(x_i, x),$$

for all $x \in X$; that is, $\bar{x}$ is an order-theoretic supremum of $(x_i)_{i \in I}$ and every $a(-, x) : X \to \mathcal{V}^{\mathrm{op}}$ preserves this supremum. Such a supremum is called ***conical supremum***.

Recall that a $\mathcal{V}$-copowered $\mathcal{V}$-category (X, a) can be viewed as an ordered set X with an action $\otimes : X \times \mathcal{V} \to X$. In terms of this structure, (X, a) has all conical suprema of a given shape I if and only if every family $(x_i)_{i \in I}$ has a supremum in the ordered set X and, moreover,

$$\left(\bigvee_{i \in I} x_i \right) \otimes u \simeq \bigvee_{i \in I} (x_i \otimes u)$$

for all $u \in \mathcal{V}$. This follows from the facts that $\bigvee_{i \in I} x_i \otimes -$ is left adjoint to $a(\bigvee_{i \in I} x_i, -)$ and

$$\mathcal{V} \xrightarrow{\Delta_{\mathcal{V}}} \mathcal{V}^I \xrightarrow{\prod_{i \in I}(x_i \otimes -)} X^I \xrightarrow{\bigvee} X$$

is left adjoint to

$$X \xrightarrow{\Delta_X} X^I \xrightarrow{\prod_{i \in I} a(x_i, -)} \mathcal{V}^I \xrightarrow{\bigwedge} \mathcal{V}.$$

A $\mathcal{V}$-category X is called ***cocomplete*** if every weighted colimit diagram has a colimit in X. One can show that X is cocomplete if and only if X has the two types of colimits described above, in this case the colimit of an arbitrary diagram can be calculated as

$$x_0 = \bigvee_{z \in A} h(z) \otimes \psi(z).$$

In particular, the $\mathcal{V}$-category $\mathcal{V}$ is cocomplete, and so are all its powers $\mathcal{V}^S$.

A $\mathcal{V}$-functor $f : X \to Y$ between cocomplete $\mathcal{V}$-categories is called ***cocontinuous*** whenever f preserves all colimits of weighted colimit diagrams; by the above, f is cocontinuous if and only if f preserves copowers and order-theoretic suprema.

Definition 2.9. A $\mathcal{V}$-category X is called ***finitely cocomplete*** whenever every weighted colimit diagram given by $h : A \to X$ and $\psi : A \mathrel{\circ\!\!\to} G$ where the underlying set of A is finite has a colimit in X. We call a $\mathcal{V}$-functor $f : X \to Y$ between finitely cocomplete $\mathcal{V}$-categories ***finitely cocontinuous*** whenever those colimits are preserved.

Therefore:

- X is finitely cocomplete if and only if X has all copowers, a bottom element, all order-theoretic binary suprema and, moreover, these suprema are preserved by all $\mathcal{V}$-functors $a(-, x) : X \to \mathcal{V}^{\mathrm{op}}$.
- A map $f : X \to Y$ between finitely cocomplete $\mathcal{V}$-categories is a finitely cocontinuous $\mathcal{V}$-functor if and only if f is monotone and preserves copowers and binary suprema. Note that, by Remark 2.6, the preservation of copowers guarantees the preservation of the bottom element.

In the sequel we write $\mathcal{V}\text{-}\mathsf{FinSup}$ to denote the category of separated finitely cocomplete $\mathcal{V}$-categories and finitely cocontinuous $\mathcal{V}$-functors. We also recall that the inclusion functor $\mathcal{V}\text{-}\mathsf{FinSup} \to \mathcal{V}\text{-}\mathsf{Cat}$ is monadic; in particular, $\mathcal{V}\text{-}\mathsf{FinSup}$ is complete and $\mathcal{V}\text{-}\mathsf{FinSup} \to \mathcal{V}\text{-}\mathsf{Cat}$ preserves all limits.

Remark 2.10. By the considerations of this section, $\mathcal{V}$-FinSup can also be seen as a quasivariety (for more information on algebraic categories we refer to [Adámek and Rosický, 1994] and [Adámek *et al.*, 2010]). In fact, a separated finitely cocomplete $\mathcal{V}$-category can be described as a set X equipped with a nullary operation $\perp$, a binary operation $\vee$, and unary operations $-\otimes u$ ($u \in \mathcal{V}$), subject to the following equations and implications:

$$x \vee x = x, \qquad x \vee y = y \vee x, \qquad x \vee (y \vee z) = (x \vee y) \vee z, \qquad x \vee \perp = x,$$
$$x \otimes k = x, \quad (x \otimes u) \otimes v = x \otimes (u \otimes v), \qquad \perp \otimes u = \perp, \qquad (x \vee y) \otimes u = (x \otimes u) \vee (y \otimes u);$$

for all $x, y, z \in X$ and $u, v \in \mathcal{V}$. We also have to impose the conditions

$$x \otimes v = \bigvee_{u \in S} (x \otimes u),$$

for all $x \in X$ and $S \subseteq \mathcal{V}$ with $v = \bigvee S$; however, this is not formulated using just the operations above. Writing $x \leq y$ as an abbreviation for the equation $y = x \vee y$, we can express the condition "$x \otimes v$ is the supremum of $\{x \otimes u \mid u \in S\}$" by the covert equational conditions

$$x \otimes u \leq x \otimes v, \qquad (u \in S)$$

and the implication

$$\bigwedge_{u \in S} (x \otimes u \leq y) \implies (x \otimes v \leq y).$$

Furthermore, the morphisms of $\mathcal{V}$-FinSup correspond precisely to the maps preserving these operations. By the considerations above, with λ denoting the smallest regular cardinal larger than the size of $\mathcal{V}$, the category $\mathcal{V}$-FinSup is equivalent to a λ-ary quasivariety. From that we conclude that $\mathcal{V}$-FinSup is also cocomplete. Finally, if the quantale $\mathcal{V}$ is based on the lattice $[0, 1]$, then it is enough to consider countable subsets $S \subseteq \mathcal{V}$, and therefore $\mathcal{V}$-FinSup is equivalent to a $\aleph_1$-ary quasivariety.

Another important class of colimit weights is the class of all right adjoint $\mathcal{V}$-distributors $\psi : A \,{\circ\!\!\to}\, G$.

Definition 2.11. A $\mathcal{V}$-category X is called ***Cauchy-complete*** whenever every diagram $(h : A \to X, \psi : A \,{\circ\!\!\to}\, G)$ with ψ right adjoint has a colimit in X.

The designation "Cauchy-complete" has its roots in Lawvere's amazing observation that, for metric spaces interpreted as $[0, \infty]$-categories, this notion coincides with the classical notion of Cauchy-completeness (see [Lawvere, 1973]). We hasten to remark that every $\mathcal{V}$-functor preserves colimits weighted by right adjoint $\mathcal{V}$-distributors.

In this context, [Hofmann and Tholen, 2010] introduces a closure operator $\overline{(-)}$ on $\mathcal{V}$-Cat which facilitates working with Cauchy-complete $\mathcal{V}$-categories. As usual, a subset $M \subseteq X$ of a $\mathcal{V}$-category (X, a) is ***closed*** whenever $M = \overline{M}$ and M is ***dense*** in X whenever $\overline{M} = X$. Below we recall the relevant facts about this closure operator.

Theorem 2.12. *The following assertions hold.*

1. *For every $\mathcal{V}$-category (X, a), $x \in X$ and $M \subseteq X$, $x \in \overline{M} \iff k \leq \bigvee_{z \in M} a(x, z) \otimes a(z, x)$.*
2. *If $\mathcal{V}$ is completely distributive (see [Raney, 1952] and [Wood, 2004]) with totally below relation $\ll$ and $k \leq \bigvee_{u \ll k} u \otimes u$, then $x \in \overline{M}$ if and only if, for every $u \ll k$, there is some $z \in M$ with $u \leq a(x, z)$ and $u \leq a(z, x)$. By [Flagg, 1992, Theorem 1.12], the quantale $\mathcal{V}$ satisfies $k \leq \bigvee_{u \ll k} u \otimes u$ provided that the subset $A = \{u \in \mathcal{V} \mid u \ll k\}$ of $\mathcal{V}$ is directed.*
3. *The $\mathcal{V}$-category $\mathcal{V}$ is Cauchy-complete.*
4. *The full subcategory of $\mathcal{V}$-Cat defined by all Cauchy-complete $\mathcal{V}$-categories is closed under limits in $\mathcal{V}$-Cat.*
5. *Let X be a Cauchy-complete and separated $\mathcal{V}$-category and $M \subseteq X$. Then the $\mathcal{V}$-subcategory M of X is Cauchy-complete if and only if the subset $M \subseteq X$ is closed in X.*

The notion of weighted colimit is dual to the one of weighted limit, of the latter we only need the special case of u-powers, with $u \in \mathcal{V}$.

Definition 2.13. A $\mathcal{V}$-category (X, a) is called ***$\mathcal{V}$-powered*** whenever, for every $x \in X$, the $\mathcal{V}$-functor $a(-, x) : (X, a)^{\mathrm{op}} \to (\mathcal{V}, \hom)$ has a left adjoint in $\mathcal{V}$-Cat.

Elementwise, this amounts to saying that, for every $x \in X$ and every $u \in \mathcal{V}$, there is an element $x \pitchfork u \in X$, called the ***u-power*** of x, satisfying

$$\hom(u, a(y, x)) = a(y, x \pitchfork u),$$

for all $y \in X$. The $\mathcal{V}$-category $\mathcal{V}$ is $\mathcal{V}$-powered where $w \pitchfork u = \hom(u, w)$, more generally, $\mathcal{V}^S$ is $\mathcal{V}$-powered with $(h \pitchfork u)(x) = \hom(u, h(x))$, for all $h \in \mathcal{V}^S$, $u \in \mathcal{V}$ and $x \in S$.

Remark 2.14. For every $\mathcal{V}$-functor $f : X \to Y$, $x \in X$ and $u \in \mathcal{V}$, $f(u \pitchfork x) \leq u \pitchfork f(x)$.

3. Continuous quantale structures on the unit interval

In this paper we are particularly interested in quantales based on the complete lattice $[0, 1]$. We succinctly review the classification of all *continuous* quantale structures $\otimes : [0, 1] \times [0, 1] \to [0, 1]$ on $[0, 1]$ with neutral element 1. Such quantale structures are also called ***continuous t-norms***. The results obtained in [Faucett, 1955] and [Mostert and Shields, 1957] show that every such tensor is a combination of the three structures mentioned in Examples 2.3(3). A more detailed presentation of this material is in [Alsina *et al.*, 2006].

We start by recalling some standard notation. An element $x \in [0, 1]$ is called ***idempotent*** whenever $x \otimes x = x$ and ***nilpotent*** whenever $x \neq 0$ and, for some $n \in \mathbb{N}$, $x^n = 0$. The number of idempotent and nilpotent elements characterises the three tensors $\wedge$, $\odot$ and $\otimes$ on $[0, 1]$ among all continuous t-norms.

Proposition 3.1. *Assume that* 0 *and* 1 *are the only idempotent elements of* $[0, 1]$ *with respect to a given continuous t-norm. Then*

(1) $[0, 1]$ *has no nilpotent elements, then* $\otimes = *$ *is multiplication.*
(2) $[0, 1]$ *has a nilpotent element, then* $\otimes = \odot$ *is the Łukasiewicz tensor. In this case, every element* x *with* $0 < x < 1$ *is nilpotent.*

To deal with the general case, for a continuous t-norm $\otimes$ consider the subset $E = \{x \in [0, 1] \mid x \text{ is idempotent}\}$. Note that E is closed in $[0, 1]$ since it can be presented as an equaliser of the diagram

$$[0,1] \underset{-\otimes-}{\overset{\text{identity}}{\rightrightarrows}} [0,1]$$

in $\mathsf{CompHaus}$.

Lemma 3.2. *Let* $\otimes$ *be a continuous t-norm on* $[0, 1]$*,* $x, y \in [0, 1]$ *and* $e \in E$ *so that* $x \leq e \leq y$*. Then* $x \otimes y = x$*.*

Corollary 3.3. *Let* $\otimes$ *be a continuous t-norm on* $[0, 1]$ *so that every element is idempotent. Then* $\otimes = \wedge$*.*

Before announcing the main result of this section, we note that, for idempotents $e < f$ in $[0, 1]$, the closed interval $[e, f]$ is a quantale with tensor defined by the restriction of the tensor on $[0, 1]$ and neutral element f.

Theorem 3.4. *Let* $\otimes$ *be a continuous t-norm on* $[0, 1]$*. For every non-idempotent* $x \in [0, 1]$*, there exist idempotent elements* $e, f \in [0, 1]$*, with* $e < x < f$*, such that the quantale* $[e, f]$ *is isomorphic to the quantale* $[0, 1]$ *with either multiplication or Łukasiewicz tensor.*

Proof. See [Mostert and Shields, 1957, Theorem B]. $\square$

Remark 3.5. We note that every isomorphism $[e, f] \to [0, 1]$ of quantales is necessarily a homeomorphism.

The following consequence of Theorem 3.4 will be useful in the sequel.

Corollary 3.6. *Let $(u, v) \in [0,1] \times [0,1]$ with $u \otimes v = 0$. Then either $u = 0$ or $v^n = 0$, for some $n \in \mathbb{N}$. Hence, if there are no nilpotent elements, then $u = 0$ or $v = 0$.*

Proof. Assume $u > 0$. The assertion is clear if there is some idempotent e with $0 < e \leq u$. If there is no $e \in E$ with $0 < e \leq u$, then there is some $f \in E$ with $u < f$ and $[0, f]$ is isomorphic to $[0,1]$ with either multiplication or Łukasiewicz tensor. Since $u \otimes v = 0$, $v < f$. If $[0, f]$ is isomorphic to $[0,1]$ with multiplication, then $v = 0$; otherwise there is some $n \in \mathbb{N}$ with $v^n = 0$. □

In conclusion, the results of this section show that every continuous t-norm on $[0,1]$ is obtained as a combination of infimum, multiplication and Łukasiewicz tensor. Conversely, continuous quantale structures on $[0,1]$ can be defined piecewise using these three elementary t-norms; for more information see [Alsina *et al.*, 2006, Theorem 2.4.2].

4. Ordered compact spaces and Vietoris monads

The Vietoris monad on the category of separated ordered compact spaces plays a key role in the duality results presented beginning from Section 6. We recall from [Nachbin, 1950, 1965] that an ***ordered compact space*** consists of a compact space X equipped with an order relation $\leq$ so that

$$\{(x, y) \mid x \leq y\} \subseteq X \times X$$

is a closed subset of the product space $X \times X$. We denote by $\mathsf{Ord_sComp}$ the category of separated ordered compact spaces and monotone continuous maps.

Remark 4.1. It follows immediately from the definition that every separated ordered compact space is Hausdorff since, for a separated order relation, the diagonal

$$\Delta = \{(x, y) \mid x \leq y\} \cap \{(x, y) \mid y \leq x\}$$

is a closed subset of $X \times X$.

Remark 4.2. There is a close connection between separated ordered compact spaces and a certain type of sober topological spaces, the so-called stably compact spaces, which was first exposed in [Gierz *et al.*, 1980]. In fact, the category $\mathsf{Ord_sComp}$ is isomorphic to the category StComp of stably compact spaces and proper maps, for details we refer to [Gierz *et al.*, 2003].

Given a separated ordered compact space X, keeping its topology but taking now its dual order produces also a separated ordered compact space, denoted by X^{op}. Of particular interest to us are the separated ordered compact space $[0,1]$ with the Euclidean topology and the usual "less or equal" relation, and its dual separated ordered compact space $[0,1]^{\mathrm{op}}$. For a subset A of X, we denote the ***up-closure*** of A by

$$\uparrow A = \{y \in X \mid y \geq x \text{ for some } x \in A\}$$

and the ***down-closure*** of A by

$$\downarrow A = \{y \in X \mid y \leq x \text{ for some } x \in A\}.$$

To distinguish between order closed and topological closed sets we say that an up-closed set is ***upper*** and a down-closed set is ***lower***. Below we collect some facts about these structures which can be found in, or follow from, [Nachbin, 1965, Proposition 4 and Theorems 1 and 4].

Proposition 4.3. *If A is a compact subset of a separated ordered compact space X then the sets $\uparrow A$ and $\downarrow A$ are closed.*

Corollary 4.4. *Let A be a subset of a separated ordered compact space X. Then, $\uparrow \overline{A} \subseteq X$ is the smallest closed upper subset containing A.*

Proposition 4.5 (Urysohn lemma)**.** *Let A and B be subsets of a separated ordered compact space X such that A is a closed upper set, B is a closed lower set and $A \cap B = \varnothing$. Then there exists a continuous and monotone function $\psi : X \to [0,1]$ such that $\psi(x) = 1$ for every $x \in A$, and $\psi(x) = 0$ for every $x \in B$.*

These results imply immediately:

Proposition 4.6. *The separated ordered compact space $[0,1]$ is an initial cogenerator in $\mathsf{Ord_sComp}$; that is, for every separated ordered compact space X, the cone $(\psi : X \to [0,1])_\psi$ of all morphisms from X to $[0,1]$ is point-separating and initial with respect to the canonical forgetful functor $\mathsf{Ord_sComp} \to \mathsf{Set}$ (see [Tholen, 2009; Hofmann and Nora, 2015] for a description of initial cones in $\mathsf{Ord_sComp}$). Since $[0,1] \simeq [0,1]^{\mathrm{op}}$ in $\mathsf{Ord_sComp}$, also $[0,1]^{\mathrm{op}}$ is an initial cogenerator in $\mathsf{Ord_sComp}$.*

The ***Vietoris functor*** $V : \mathsf{Ord_sComp} \to \mathsf{Ord_sComp}$ sends a separated ordered compact space X to the space VX of all closed upper subsets of X, with order containment $\supseteq$, and the compact topology is generated by the sets

$$\{A \subseteq X \mid A \text{ closed upper and } A \cap U \neq \varnothing\} \quad (U \subseteq X \text{ open lower}),$$
$$\{A \subseteq X \mid A \text{ closed upper and } A \cap K = \varnothing\} \quad (K \subseteq X \text{ closed lower}).$$

Given a map $f : X \to Y$ in $\mathsf{Ord_sComp}$, the functor returns the map that sends a closed upper subset $A \subseteq X$ to the up-closure $\uparrow f[A]$ of $f[A]$. The Vietoris functor $V : \mathsf{Ord_sComp} \to \mathsf{Ord_sComp}$ is part of a monad $\mathbb{V} = (V, e, m)$ with unit and multiplication defined by

$$e_X : X \longrightarrow VX,\ x \longmapsto \Uparrow\{x\} \qquad \text{and} \qquad m_X : VVX \longrightarrow VX,\ \mathcal{A} \longmapsto \bigcup \mathcal{A}.$$

For more information about this construction (in the context of topological spaces, see Remark 4.2) we refer to [Schalk, 1993, Section 6.3].

Furthermore, from the monad $\mathbb{V}$ on $\mathsf{Ord_sComp}$ we obtain a monad $\mathbb{V} = (V, m, e)$ on the category $\mathsf{CompHaus}$ of compact Hausdorff spaces and continuous maps via the canonical adjunction

$$\begin{tikzcd} \mathsf{Ord_sComp} \arrow[r, bend left, "\text{forgetful}"] \arrow[r, phantom, "\top"] & \mathsf{CompHaus}. \arrow[l, bend left, "\text{discrete}"] \end{tikzcd}$$

The functor $V : \mathsf{CompHaus} \to \mathsf{CompHaus}$ sends a compact Hausdorff space X to the space

$$VX = \{A \subseteq X \mid A \text{ is closed}\}$$

with the topology generated by the sets

$$\{A \in VX \mid A \cap U \neq \varnothing\} \quad (U \subseteq X \text{ open}) \qquad \text{and} \qquad \{A \in VX \mid A \cap K = \varnothing\} \quad (K \subseteq X \text{ closed});$$

for $f : X \to Y$ in $\mathsf{CompHaus}$, $Vf : VX \to VY$ sends A to $f[A]$. We note that this is indeed the original construction introduced by Vietoris in [Vietoris, 1922].

Remark 4.7. In this paper we are interested in the Kleisli categories $\mathsf{Ord_sComp}_{\mathbb{V}}$ and $\mathsf{CompHaus}_{\mathbb{V}}$. A morphism $X \to VY$ in $\mathsf{CompHaus}$ corresponds to a relation from X to Y, written as $X \relbar\!\!\!\!\!\!\nrightarrow Y$. Likewise, a morphism $X \to VY$ in $\mathsf{Ord_sComp}$ corresponds to a distributor between the underlying separated ordered sets, and we write $X \,{\circ\!\!\!\rightarrow}\, Y$ instead of $X \to VY$. Furthermore, in both cases composition in the Kleisli category corresponds to relational composition. Also note that $\mathsf{CompHaus}_{\mathbb{V}}$ is isomorphic to the full subcategory of $\mathsf{Ord_sComp}_{\mathbb{V}}$ determined by the discretely ordered compact spaces.

5. Dual adjunctions

In this section we present some well-known results about the structure and construction of dual adjunctions. There is a vast literature on this subject, we mention here [Lambek and Rattray, 1978, 1979], [Dimov and Tholen, 1989], [Porst and Tholen, 1991], [Johnstone, 1986] and [Clark and Davey, 1998].

We start by considering an adjunction

$$\begin{tikzcd} \mathsf{X} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{A}^{\mathrm{op}} \arrow[l, bend left, "G"] \end{tikzcd} \tag{5.i}$$

between a category X and the dual of a category A. In general, such an adjunction is not an equivalence. Nevertheless, one can always consider its restriction to the full subcategories $\mathrm{Fix}(\eta)$ and $\mathrm{Fix}(\varepsilon)$ of X respectively A, defined by the classes of objects

$$\{X \mid \eta_X \text{ is an isomorphism}\} \qquad \text{and} \qquad \{A \mid \varepsilon_A \text{ is an isomorphism}\},$$

where it yields an equivalence $\mathrm{Fix}(\eta) \simeq \mathrm{Fix}(\varepsilon)^{\mathrm{op}}$. The passage from X to $\mathrm{Fix}(\eta)$ is only useful if this subcategory contains all "interesting objects". This, however, is not always the case; $\mathrm{Fix}(\eta)$ can be even empty. *En passant* we mention that these fixed subcategories are reflective in A respectively in X provided that the monad induced by the adjunction (5.i) on A respectively X is idempotent (see [Lambek and Rattray, 1979, Theorem 2.0] for details).

Throughout this section we assume that X and A are equipped with faithful functors

$$|-| : \mathsf{X} \longrightarrow \mathsf{Set} \qquad \text{and} \qquad |-| : \mathsf{A} \longrightarrow \mathsf{Set}.$$

Definition 5.1. The adjunction (5.i) is ***induced by the dualising object*** $(\widetilde{X}, \widetilde{A})$, with objects $\widetilde{X}$ in X and $\widetilde{A}$ in A, whenever $|\widetilde{X}| = |\widetilde{A}|$, $|F| = \hom(-, \widetilde{X})$, $|G| = \hom(-, \widetilde{A})$ and the units are given by

$$\begin{aligned} \eta_X : X &\longrightarrow GFX \\ x &\longmapsto \mathrm{ev}_x \end{aligned} \qquad \text{and} \qquad \begin{aligned} \varepsilon_A : A &\longrightarrow FGA; \\ a &\longmapsto \mathrm{ev}_a \end{aligned} \tag{5.ii}$$

with ev_x and ev_a denoting the evaluation maps.

If the forgetful functors to Set are representable by objects X_0 in X and A_0 in A, then every adjunction (5.i) is of this form, up to natural equivalence (see [Dimov and Tholen, 1989] and [Porst and Tholen, 1991]).

Remark 5.2. Consider an adjunction (5.i) induced by a dualising object $(\widetilde{X}, \widetilde{A})$. For every $\psi : X \to \widetilde{X}$ and $\varphi : A \to \widetilde{A}$, the diagrams

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\psi"'] & GFX \arrow[d, "\mathrm{ev}_\psi"] \\ & \widetilde{X} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \arrow[r, "\varepsilon_A"] \arrow[dr, "\varphi"'] & FGA \arrow[d, "\mathrm{ev}_\varphi"] \\ & \widetilde{A} \end{tikzcd}$$

commute.

We turn now to the question "How to construct dual equivalences?". Motivated by the considerations above, we assume that $\widetilde{X}$ and $\widetilde{A}$ are objects in X and A respectively, with the same underlying set $|\widetilde{X}| = |\widetilde{A}|$. In order to obtain a dual adjunction, we wish to lift the hom-functors $\hom(-, \widetilde{X}) : \mathsf{X}^{\mathrm{op}} \to \mathsf{Set}$ and $\hom(-, \widetilde{A}) : \mathsf{A}^{\mathrm{op}} \to \mathsf{Set}$ to functors $F : \mathsf{X}^{\mathrm{op}} \to \mathsf{A}$ and $G : \mathsf{A}^{\mathrm{op}} \to \mathsf{X}$ in such a way that the maps (5.ii) underlie an X-morphism and, respectively, an A-morphism. To this end, we consider the following two conditions.

(Init X): For each object X in X, the cone $(\mathrm{ev}_x : \hom(X, \widetilde{X}) \to |\widetilde{A}|, \psi \mapsto \psi(x))_{x \in |X|}$ admits an initial lift $(\mathrm{ev}_x : F(X) \to \widetilde{A})_{x \in |X|}$.

(Init A): For each object A in A, the cone $(\mathrm{ev}_a : \hom(A, \widetilde{A}) \to |\widetilde{X}|))_{a \in |A|}$ admits an initial lift $(\mathrm{ev}_a : G(A) \to \widetilde{X})_{b \in V(B)}$.

Theorem 5.3. *If conditions (Init X) and (Init A) are fulfilled, then these initial lifts define the object parts of a dual adjunction* (5.i) *induced by* $(\widetilde{X}, \widetilde{A})$.

Clearly, if $|-|:\mathsf{X}\to\mathsf{Set}$ is topological (see [Adámek *et al.*, 1990]), then (Init X) is fulfilled. The following proposition describes another typical situation.

Proposition 5.4. *Let* A *be the category of algebras for a signature* Ω *of operation symbols and assume that* X *is complete and* $|-|:\mathsf{X}\to\mathsf{Set}$ *preserves limits. Furthermore, assume that, for every operation symbol* $\omega\in\Omega$*, the corresponding operation* $|\widetilde{A}|^I\to|\widetilde{A}|$ *underlies an* X*-morphism* $\widetilde{X}^I\to\widetilde{X}$*. Then both (Init X) and (Init A) are fulfilled.*

Proof. This result is essentially proven in [Lambek and Rattray, 1979, Proposition 2.4]. Firstly, since all operations on $\widetilde{A}$ are X-morphisms, the algebra structure on $\hom(X,\widetilde{X})$ can be defined pointwise. Secondly, for each algebra A, the canonical inclusion $\hom(A,\widetilde{A})\to|\widetilde{X}|^{|A|}$ is the equaliser of a pair of X-morphisms between powers of $\widetilde{X}$. In fact, a map $f:|A|\to|\widetilde{A}|$ is an algebra homomorphism whenever, for every operation symbol $\omega\in\Omega$ with arity I and every $h\in|A|^I$,

$$f(\omega_A(h))=\omega_{\widetilde{A}}(f\cdot h).$$

In other words, the set of maps $f:|A|\to|\widetilde{A}|$ which preserve the operation ω is precisely the equaliser of

$$\pi_{\omega_A(h)}:|\widetilde{A}|^{|A|}\longrightarrow|\widetilde{A}|$$

and the composite

$$|\widetilde{A}|^{|A|}\xrightarrow{-\cdot h}|\widetilde{A}|^{I}\xrightarrow{\omega_{\widetilde{A}}}|\widetilde{A}|.$$

Since both maps underlie X-morphisms $\widetilde{X}^{|A|}\to\widetilde{X}$, the assertion follows. □

Remark 5.5. The result above remains valid if

- the objects of A admit an order relation and some of the operations are only required to be preserved laxly, and
- the order relation $R\to|\widetilde{A}|\times|\widetilde{A}|$ of $\widetilde{A}$ underlies an X-morphism $R'\to\widetilde{X}\times\widetilde{X}$.

In fact, with the notation of the proof above, the set of maps $f:|A|\to|\widetilde{A}|$ with

$$f(\omega_A(h))\le\omega_{\widetilde{A}}(f\cdot h)$$

for all $h\in|A|^I$ can be described as the pullback of the diagram

$$\begin{tikzcd} & R \arrow[d] \\ |\widetilde{A}|^{|A|} \arrow[r] & |\widetilde{A}|\times|\widetilde{A}|. \end{tikzcd}$$

Clearly, for every object X in X, the unit $\eta_X:X\to GF(X)$ is an isomorphism if and only if η_X is surjective and an embedding. If the dual adjunction is constructed using (Init X) and (Init A), then, by Remark 5.2,

η_X is an embedding if and only if the cone $(\psi:X\to\widetilde{X})_\psi$ is point-separating and initial.

We hasten to remark that the latter condition only depends on $\widetilde{X}$ and is independent of the choice of A. If η is not componentwise an embedding, we can substitute X by its full subcategory defined by all those objects X where $(\psi:X\to\widetilde{X})_\psi$ is point-separating and initial; by construction, the functor G corestricts to this subcategory. Again, this procedure is only useful if this subcategory contains all "interesting spaces", otherwise it is probably best to use a different dualising object. For exactly this reason, in this paper we will consider the compact Hausdorff space $[0,1]$ instead of the discrete two-element space.

We assume now that η is componentwise an embedding. Then the functor $F:\mathsf{X}\to\mathsf{A}^{\mathrm{op}}$ is faithful, and η is an isomorphism if and only if F is also full. Put differently, if η is not an isomorphism, then A has too many arrows. A possible way to overcome this problem is to enrich the structure of A. For instance, in [Johnstone, 1986, VI.4.4] it is shown that, under mild conditions, A can be substituted by the category of Eilenberg–Moore algebras for the monad on A induced by the dual of the adjunction (5.i). However, in this paper we take a different approach: instead of saying "A has too many morphisms", one might also think that "X has too few morphisms". One way of adding morphisms to a category is passing from X to the Kleisli category $\mathsf{X}_{\mathbb{T}}$, for a suitable monad $\mathbb{T}$ on X. In fact, and rather trivially, for $\mathbb{T}$ being the monad on X induced by the adjunction (5.i), the comparison functor $\mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$ is fully faithful. In general, this procedure will not give us new insights since we do not know much about the monad induced by $F \dashv G$. The situation improves if we take a different, better known monad $\mathbb{T}$ on X *isomorphic* to the monad induced by $F \dashv G$. We are then left with the task of identifying the X-morphisms inside $\mathsf{X}_{\mathbb{T}}$ in a purely categorical way so that it can be translated across a duality.

Example 5.6. Consider the power monad $\mathbb{P}$ on Set whose Kleisli category $\mathsf{Set}_{\mathbb{P}}$ is equivalent to the category Rel of sets and relations. Within Rel, functions can be identified by two fundamentally different properties.

- A relation $r : X \to\!\!\!\!\!\!\!\!+\;\; Y$ is a function if and only if r has a right adjoint in the ordered category Rel. This is actually a 2-categorical property; if we want to use it in a duality we must make sure that the involved equivalence functors are locally monotone.
- A relation $r : X \to\!\!\!\!\!\!\!\!+\;\; Y$ is a function if and only if r is a homomorphism of comonoids in the monoidal category Rel, that is, the diagrams

$$\begin{tikzcd} X \arrow[r, "r", "\mid" marking] \arrow[dr, "\top"', "\mid" marking] & Y \arrow[d, "\top", "\mid" marking] \\ & 1 \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} X \times X \arrow[r, "r\times r", "\mid" marking] & Y \times Y \\ X \arrow[u, "\Delta", "\mid" marking] \arrow[r, "r"', "\mid" marking] & Y \arrow[u, "\Delta"', "\mid" marking] \end{tikzcd}$$

commute. In the second diagram, $X \times X$ denotes the set-theoretical product which can be misleading since it is not the categorical product in Rel. To use this description in a duality result, one needs to know the corresponding operation on the other side.

In the considerations above, the Kleisli category $\mathsf{X}_{\mathbb{T}}$ was only introduced to support the study of X; however, at some occasions our primary interest lies in $\mathsf{X}_{\mathbb{T}}$. In this case, a monad $\mathbb{T}$ on X is typically given before-hand, and we wish to find an adjunction (5.i) so that the induced monad is isomorphic to $\mathbb{T}$. If a dualising object $(\widetilde{X}, \widetilde{A})$ induces this adjunction, we speak of a ***functional representation*** of $\mathbb{T}$. Looking again at the example $\mathrm{CoAlg}(V) \simeq \mathsf{BAO}^{\mathrm{op}}$ of Section 1, by observing that V is part of a monad $\mathbb{V} = (V, m, e)$ on BooSp, we can think of the objects of $\mathrm{CoAlg}(V)$ as Boolean spaces X equipped with an endomorphism $r : X \to\!\!\!\!\!\!\!\!+\;\; X$ in $\mathsf{BooSp}_{\mathbb{V}}$; the morphisms of $\mathrm{CoAlg}(V)$ are those morphisms of BooSp commuting with this additional structure. Halmos' duality theorem [Halmos, 1956] affirms that the category $\mathsf{BooSp}_{\mathbb{V}}$ is dually equivalent to the category $\mathsf{FinSup}_{\mathrm{BA}}$ of Boolean algebras with finite suprema preserving maps. The duality $\mathrm{CoAlg}(V) \simeq \mathsf{BAO}^{\mathrm{op}}$ follows now from both Halmos duality and the classical Stone duality $\mathsf{BooSp} \simeq \mathsf{BA}^{\mathrm{op}}$ [Stone, 1936]. We note that [Halmos, 1956] does not talk about monads, but in [Hofmann and Nora, 2015] we have studied this and other dualities from this point of view.

As we explained above, our aim is to construct and analyse functors $F : \mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$ which extend a given functor $F : \mathsf{X} \to \mathsf{A}^{\mathrm{op}}$ that is part of an adjunction $F \dashv G$ induced by a dualising object $(\widetilde{X}, \widetilde{A})$. It is well-known that such functors $F : \mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$ correspond precisely to monad morphisms from $\mathbb{T}$ to the monad induced by $F \dashv G$, and that monad morphisms into a "double dualisation monad" are in bijection with certain algebra structures on $\widetilde{X}$ (see [Kock, 1971], for instance). In the remainder of this section, we explain these correspondences in the specific context of our paper.

Let X and A be categories with respresentable forgetful functors

$$|-| \simeq \hom(X_0, -) : \mathsf{X} \longrightarrow \mathsf{Set} \qquad \text{and} \qquad |-| \simeq \hom(A_0, -) : \mathsf{A} \longrightarrow \mathsf{Set},$$

$\mathbb{T} = (T, m, e)$ a monad on X and $F \dashv G$ an adjunction

$$\begin{tikzcd} \mathsf{X} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{A}^{\mathrm{op}} \arrow[l, bend left, "G"] \end{tikzcd}$$

induced by $(\widetilde{X}, \widetilde{A})$. We denote by $\mathbb{D}$ the monad induced by $F \dashv G$. The next result establishes a connection between monad morphisms $j : \mathbb{T} \to \mathbb{D}$ and $\mathbb{T}$-algebra structures on $\widetilde{X}$ *compatible* with the adjunction $F \dashv G$.

Theorem 5.7. *In the setting described above, the following data are in bijection.*

(1) Monad morphisms $j : \mathbb{T} \to \mathbb{D}$.

(2) Functors $F : \mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$ making the diagram

$$\begin{tikzcd} \mathsf{X}_{\mathbb{T}} \arrow[r, "F"] & \mathsf{A}^{\mathrm{op}} \\ \mathsf{X} \arrow[u, "F_{\mathbb{T}}"] \arrow[ur, "F"'] & \end{tikzcd}$$

commutative.

(3) $\mathbb{T}$-algebra structures $\sigma : T\widetilde{X} \to \widetilde{X}$ such that the map

$$\hom(X, \widetilde{X}) \longrightarrow \hom(TX, \widetilde{X}),\ \psi \longmapsto \sigma \cdot T\psi$$

is an A-morphism $\kappa_X : FX \to FTX$, for every object X in X.

Proof. The equivalence between the data described in (1) and (2) is well-known, see [Pumplün, 1970], for instance. We recall here that, for a monad morphism $j : \mathbb{T} \to \mathbb{D}$, the corresponding functor $F : \mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$ can be obtained as

$$\mathsf{X}_{\mathbb{T}} \xrightarrow{\text{composition with } j} \mathsf{X}_{\mathbb{D}} \xrightarrow{\text{comparison}} \mathsf{A}^{\mathrm{op}}.$$

To describe the passage from (1) to (3), we recall from [Johnstone, 1986, Lemma VI.4.4] that $\widetilde{X}$ becomes a $\mathbb{D}$-algebra since $\widetilde{X} \simeq GA_0$ and $G : \mathsf{A}^{\mathrm{op}} \to \mathsf{X}$ factors as

$$\begin{tikzcd} \mathsf{A}^{\mathrm{op}} \arrow[r, "\text{comparision}"] \arrow[dr, "G"'] & \mathsf{X}^{\mathbb{D}} \arrow[d, "\text{forgetful}"] \\ & \mathsf{X}. \end{tikzcd}$$

A little computation shows that the $\mathbb{D}$-algebra structure on $\widetilde{X}$ is

$$GF\widetilde{X} \xrightarrow{\mathrm{ev}_{1_{\widetilde{X}}}} \widetilde{X}.$$

Composing $\mathrm{ev}_{1_{\widetilde{X}}}$ with $j_{\widetilde{X}}$ gives a $\mathbb{T}$-algebra structure $\sigma : T\widetilde{X} \to \widetilde{X}$. Furthermore, the functor $F : \mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$ sends $1_{TX} : TX \nrightarrow X$ in $\mathsf{X}_{\mathbb{T}}$ to the A-morphism $Fj_X \cdot \varepsilon_{FX} : FX \to FTX$ which sends $\psi \in FX$ to $Fj_X(\mathrm{ev}_\psi) = \mathrm{ev}_\psi \cdot j_X$. On the other hand,

$$\sigma \cdot T\psi = \mathrm{ev}_{1_{\widetilde{X}}} \cdot j_{\widetilde{X}} \cdot T\psi = \mathrm{ev}_{1_{\widetilde{X}}} \cdot GF\psi \cdot j_X = \mathrm{ev}_\psi \cdot j_X;$$

which shows that $\kappa_X = Fj_X \cdot \varepsilon_{FX}$ is an A-morphism. For a compatible $\mathbb{T}$-algebra structures $\sigma : T\widetilde{X} \to \widetilde{X}$ as in (3),

$$(\varphi : X \to TY) \longmapsto (FY \xrightarrow{\kappa_Y} FTY \xrightarrow{F\varphi} FX)$$

defines a functor $F : \mathsf{X}_{\mathbb{T}} \to \mathsf{A}^{\mathrm{op}}$ making the diagram

$$\begin{tikzcd} \mathsf{X}_{\mathbb{T}} \arrow[r, "F"] & \mathsf{A}^{\mathrm{op}} \\ \mathsf{X} \arrow[u, "F_{\mathbb{T}}"] \arrow[ur, "F"'] & \end{tikzcd}$$

commutative. The induced monad morphism $j : \mathbb{T} \to \mathbb{D}$ is given by the family of maps

$$j_X : |TX| \longrightarrow \hom(FX, \widetilde{A}),\ \mathfrak{x} \longmapsto (\psi \mapsto \sigma \cdot T\psi(\mathfrak{x})).$$

Furthermore, the $\mathbb{T}$-algebra structure induced by this j is indeed

$$\mathrm{ev}_{1_{\widetilde{X}}} \cdot j_{\widetilde{X}} = \sigma \cdot T1_{\widetilde{X}} = \sigma.$$

Finally, for a monad morphism $j : \mathbb{T} \to \mathbb{D}$, the monad morphism induced by the corresponding algebra structure σ has as X-component the map sending $\mathfrak{x} \in TX$ to

$$\sigma \cdot T\psi(\mathfrak{x}) = \mathrm{ev}_\psi \cdot j_X(\mathfrak{x}) = j_X(\mathfrak{x})(\psi). \qquad \square$$

Remark 5.8. The constructions described above seem to be more natural if $\widetilde{X} = TX_0$ with $\mathbb{T}$-algebra structure m_{X_0}, see [Hofmann and Nora, 2015, Proposition 4.3]. In this case, the functor $F : \mathsf{X}_\mathbb{T} \to \mathsf{A}^{\mathrm{op}}$ is a lifting of the hom-functor $\hom(-, X_0) : \mathsf{X}_\mathbb{T} \to \mathsf{Set}^{\mathrm{op}}$. Furthermore, interpreting the elements of TX as morphisms $\varphi : X_0 \nrightarrow X$ in the Kleisli category $\mathsf{X}_\mathbb{T}$ allows to describe the components of the monad morphism j using composition in $\mathsf{X}_\mathbb{T}$:

$$j_X : |TX| \longrightarrow \hom(FX, \widetilde{A}),\ \varphi \longmapsto (\psi \mapsto \psi \cdot \varphi).$$

In Section 9 we apply this construction to a variation of the Vietoris monad on a category of "metric compact Hausdorff spaces". Unlike the classical Vietoris functor, the functor of this monad sends the one-element space to $[0,1]^{\mathrm{op}}$ (see Example 2.3 (3) for the introduction of $[0,1]$-categories as metric spaces).

6. Duality theory for continuous distributors

In this section we apply the results presented in Section 5 to the Vietoris monad $\mathbb{V}$ on $\mathsf{X} = \mathsf{Ord_sComp}$, with $\widetilde{X} = [0,1]^{\mathrm{op}}$ (see Section 4) and $\mathbb{V}$-algebra structure

$$V([0,1]^{\mathrm{op}}) \longrightarrow [0,1]^{\mathrm{op}},\ A \longmapsto \sup_{x \in A} x.$$

Then, for a category A and an adjunction

$$\begin{tikzcd} \mathsf{Ord_sComp} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] & \mathsf{A}^{\mathrm{op}} \arrow[l, bend left, "G"] \end{tikzcd}$$

induced by $([0,1]^{\mathrm{op}}, [0,1])$ and compatible with the $\mathbb{V}$-algebra structure on $[0,1]^{\mathrm{op}}$, the corresponding monad morphism j has as components the maps

$$j_X : VX \longrightarrow GC(X),\ A \longmapsto (\Phi_A : CX \to [0,1],\ \psi \mapsto \sup_{x \in A} \psi(x)).$$

We wish to find an appropriate category A so that j is an isomorphism. Our first inspiration stems from [Shapiro, 1992] where the following result is proven.

Theorem 6.1. *Consider the subfunctor $V_1 : \mathsf{CompHaus} \to \mathsf{CompHaus}$ of V sending X to the space of all non-empty closed subsets of X. The functor $V_1 : \mathsf{CompHaus} \to \mathsf{CompHaus}$ is naturally isomorphic to the functor which sends X to the space of all functions*

$$\Phi : C(X, \mathbb{R}_0^+) \longrightarrow \mathbb{R}_0^+$$

satisfying the conditions (for all $\psi, \psi_1, \psi_2 \in C(X, \mathbb{R}_0^+)$ and $u \in \mathbb{R}_0^+$)

(1) Φ is monotone,
(2) $\Phi(u * \psi) = u * \Phi(\psi)$,
(3) $\Phi(\psi_1 + \psi_2) \le \Phi(\psi_1) + \Phi(\psi_2)$,
(4) $\Phi(\psi_1 \cdot \psi_2) \le \Phi(\psi_1) \cdot \Phi(\psi_2)$,
(5) $\Phi(\psi_1 + u) = \Phi(\psi_1) + u$,
(6) $\Phi(u) = u$.

The topology on the set of all maps $\Phi : C(X, \mathbb{R}_0^+) \to \mathbb{R}_0^+$ satisfying the conditions above is the initial one with respect to all evaluation maps ev_ψ, where $\psi \in C(X, \mathbb{R}_0^+)$. The X-component of the natural isomorphism sends a closed non-empty subset $A \subseteq X$ to the map $\Phi_A : C(X, \mathbb{R}_0^+) \to \mathbb{R}_0^+$ defined by

$$\Phi_A(\psi) = \sup_{x \in A} \psi(x).$$

One notices immediately that, for $A = \varnothing$, Φ_A does not satisfy the last two axioms above. In fact, as we show below, the condition (5) is not necessary for Shapiro's result; moreover, when generalising from multiplication $*$ to an arbitrary continuous quantale structure $\otimes$ on $[0,1]$, we change $+$ in (5) to truncated minus, which is compatible with the empty set. Thanks to (2), the condition (6) can be equivalently expressed as $\Phi(1) = 1$, and this is purely related to $A \neq \varnothing$ (see Proposition 6.7) and therefore the case $A = \varnothing$ does not present any problem. To fit better into our framework, in the sequel we will consider functions into $[0,1]$ instead of $\mathbb{R}_0^+$, and also consider binary suprema $\vee$ instead of $+$ in (3).

Assumption 6.2. From now on $\otimes$ is a quantale structure on $[0,1]$ with neutral element 1. Note that then necessarily $u \otimes v \leq u \wedge v$, for all $u, v \in [0,1]$. In order to be able to combine continuous functions $\psi_1, \psi_2 : X \to [0,1]$, we assume that $\otimes : [0,1] \times [0,1] \to [0,1]$ is continuous with respect to the Euclidean topology on $[0,1]$. In other words, we consider a continuous t-norm on $[0,1]$.

Remark 6.3. There is also interesting work of Radul on a "functional representation of the Vietoris monad" in terms of functionals, notably [Radul, 1997, 2009]. In particular, Radul shows that the Vietoris monad is isomorphic to the monad defined by all real-valued "functionals which are normed, weakly additive, preserve max and weakly preserve min".

As we explain already in Section 1, in this paper we wish to introduce types of $[0,1]$-categories which resemble the order-theoretic notions of distributive lattice and Boolean algebra appearing in the classical duality theorems of Stone and Halmos. Note that a distributive lattice X is in particular a finite sup-lattice equipped with a commutative monoid structure $\wedge : X \times X \to X$ with neutral element the top-element of X and where, moreover, the maps $x \wedge - : X \to X$ preserve finite suprema. Also note that every monotone map $f : X \to Y$ between lattices laxly preserves infima, that is, $f(x \wedge x') \leq f(x) \wedge f(x')$, for all $x, x' \in X$. Below we introduce a $[0,1]$-enriched counterpart of distributive lattices where the monoid structure need not be the infimum since the tensor product on $[0,1]$ need not be the infimum; we think of these $[0,1]$-categories as *generalised* $[0,1]$-*enriched (distributive) lattices*. We recall that $[0,1]$-FinSup denotes the category of separated finitely cocomplete $[0,1]$-categories and finite colimit preserving $[0,1]$-functors; the unit interval $[0,1]$ equipped with $\hom : [0,1] \times [0,1] \to [0,1]$ is certainly an object of $[0,1]$-FinSup.

- The category
$$[0,1]\text{-}\mathsf{GLat}$$
has as objects separated finitely cocomplete $[0,1]$-categories X equipped with an associative and commutative operation $\circledcirc : X \times X \to X$ with unit element 1 which is also the top-element of X and so that the map $x \circledcirc - : X \to X$ is a finitely cocontinuous $[0,1]$-functor, for every $x \in X$; the morphisms of $[0,1]$-GLat are the finitely cocontinuous $[0,1]$-functors preserving the unit and the multiplication $\circledcirc$.
- The category
$$[0,1]\text{-}\mathsf{LaxGLat}$$
has the same objects as $[0,1]$-GLat and the morphisms are finitely cocontinuous $[0,1]$-functors $f : X \to Y$ preserving laxly the monoid structure, that is,
$$f(x \circledcirc x') \leq f(x) \circledcirc f(x'),$$
for all $x, x' \in X$.

Note that, for every $u \in [0,1]$ and $x \in X$, we have
$$x \circledcirc (1 \otimes u) = (x \circledcirc 1) \otimes u = x \otimes u,$$
therefore we can think of $\circledcirc : X \times X \to X$ as an "extension" of $\otimes : X \times [0,1] \to X$ and write $x \otimes x'$ instead of $x \circledcirc x'$. Thinking more in algebraic terms, $[0,1]$-GLat is a $\aleph_1$-ary quasivariety; in fact, by adding to the algebraic theory of $[0,1]$-FinSup (see Remark 2.10) the operations and equations describing the

monoid structure, one obtains a presentation by operations and implications. It follows in particular that $[0,1]$-GLat is complete and cocomplete. The forgetful functor

$$[0,1]\text{-}\mathsf{GLat} \longrightarrow [0,1]\text{-}\mathsf{FinSup}$$

preserves limits, and it is easy to see that the faithful functor

$$[0,1]\text{-}\mathsf{GLat} \longrightarrow [0,1]\text{-}\mathsf{LaxGLat}$$

preserves limits as well.

In the sequel we consider the $[0,1]$-category $[0,1]$ as an object of $[0,1]$-GLat with multiplication given by the tensor product $\otimes : [0,1] \times [0,1] \to [0,1]$ of $[0,1]$. Note that $\otimes : [0,1]^{\mathrm{op}} \times [0,1]^{\mathrm{op}} \to [0,1]^{\mathrm{op}}$ is a morphism in $\mathsf{Ord}_{\mathrm{s}}\mathsf{Comp}$. From Theorem 5.3 and Remark 5.5, we obtain:

Proposition 6.4. *The dualising object* $([0,1]^{\mathrm{op}}, [0,1])$ *induces a natural dual adjunction*

$$\begin{tikzcd} \mathsf{Ord}_{\mathrm{s}}\mathsf{Comp} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] & {[0,1]\text{-}\mathsf{LaxGLat}^{\mathrm{op}}.} \arrow[l, bend left, "G"] \end{tikzcd}$$

Here CX is given by $\mathsf{Ord}_{\mathrm{s}}\mathsf{Comp}(X,[0,1]^{\mathrm{op}})$ *with all operations defined pointwise, and GA is the space* $[0,1]$-$\mathsf{LaxGLat}(A,[0,1])$ *equipped with the initial topology with respect to all evaluation maps*

$$\mathrm{ev}_a : [0,1]\text{-}\mathsf{LaxGLat}(A,[0,1]) \longrightarrow [0,1]^{\mathrm{op}},\ \Phi \longmapsto \Phi(a).$$

Proof. In terms of the algebraic presentation of the $[0,1]$-category $[0,1]$ of Remark 2.10, the operations $\vee$ and $-\otimes u$ are morphisms $\vee : [0,1]^{\mathrm{op}} \times [0,1]^{\mathrm{op}} \to [0,1]^{\mathrm{op}}$ and $-\otimes u : [0,1]^{\mathrm{op}} \to [0,1]^{\mathrm{op}}$ in $\mathsf{Ord}_{\mathrm{s}}\mathsf{Comp}$, and the order relation of $[0,1]^{\mathrm{op}}$ is closed in $[0,1]^{\mathrm{op}} \times [0,1]^{\mathrm{op}}$. Therefore the assertion follows from Theorem 5.3 and Proposition 5.4. ☐

The separated ordered compact space $[0,1]^{\mathrm{op}}$ is a $\mathbb{V}$-algebra with algebra structure $\sup : V([0,1]^{\mathrm{op}}) \to [0,1]^{\mathrm{op}}$, and one easily verifies that

$$\hom(X,[0,1]^{\mathrm{op}}) \longrightarrow \hom(VX,[0,1]^{\mathrm{op}}),\ \psi \longmapsto (A \mapsto \sup_{x\in A} \psi(x))$$

is a morphism $CX \to CVX$ in $[0,1]$-$\mathsf{LaxGLat}$. By Theorem 5.7 and Remark 5.8, we obtain a commutative diagram

$$\begin{tikzcd} \mathsf{Ord}_{\mathrm{s}}\mathsf{Comp}_{\mathbb{V}} \arrow[rr, "C"] & & {[0,1]\text{-}\mathsf{LaxGLat}^{\mathrm{op}},} \\ & \mathsf{Ord}_{\mathrm{s}}\mathsf{Comp} \arrow[ul] \arrow[ur, "C"'] & \end{tikzcd}$$

of functors; where, for $\varphi : X \to\!\!\!\!\circ\ Y$ in $\mathsf{Ord}_{\mathrm{s}}\mathsf{Comp}_{\mathbb{V}}$,

$$\begin{aligned} C\varphi : CY &\longrightarrow CX \\ \psi &\longmapsto \left(x \mapsto \sup_{x\,\varphi\, y} \psi(y)\right). \end{aligned}$$

The induced monad morphism j is precisely given by the family of maps

$$j_X : VX \longrightarrow [0,1]\text{-}\mathsf{LaxGLat}(CX,[0,1]),\ A \longmapsto \Phi_A,$$

with

$$\Phi_A : CX \longrightarrow [0,1],\ \psi \longmapsto \sup_{x\in A} \psi(x).$$

In order to show that j is an isomorphism, it will be convenient to refer individually to the components of the structure of CX; that is, we consider the following conditions on a map $\Phi : CX \to [0,1]$.

(Mon): Φ is monotone.
(Act): For all $u \in [0,1]$ and $\psi \in CX$, $\Phi(u \otimes \psi) = u \otimes \Phi(\psi)$.
(Sup): For all $\psi_1, \psi_2 \in CX$, $\Phi(\psi_1 \vee \psi_2) = \Phi(\psi_1) \vee \Phi(\psi_2)$.

(Ten)$_{\mathrm{lax}}$: For all $\psi_1, \psi_2 \in CX$, $\Phi(\psi_1 \otimes \psi_2) \leq \Phi(\psi_1) \otimes \Phi(\psi_2)$.
(Ten): For all $\psi_1, \psi_2 \in CX$, $\Phi(\psi_1 \otimes \psi_2) = \Phi(\psi_1) \otimes \Phi(\psi_2)$.
(Top): $\Phi(1) = 1$.

Remark 6.5. The condition (Act) implies $\Phi(0) = 0$ and (Sup) and implies (Mon). Also note that, by (Mon) and (Act), if $\psi(x) \leq u$ for all $x \in X$, then $\Phi(\psi) \leq u$. Finally, if $\otimes = \wedge$, then (Ten)$_{\mathrm{lax}}$ is a consequence of (Mon).

Clearly, $\Phi : CX \to [0,1]$ is a morphism in $[0,1]$-$\mathsf{LaxGLat}$ if and only if Φ satisfies the conditions (Mon), (Act), (Sup) and (Ten)$_{\mathrm{lax}}$; and $\Phi : CX \to [0,1]$ is a morphism in $[0,1]$-GLat if and only if Φ satisfies the conditions (Mon), (Act), (Sup), (Top) and (Ten).

Definition 6.6. A closed upper subset $A \subseteq X$ of a separated ordered compact space is called ***irreducible*** whenever, for all closed upper subsets $A_1, A_2 \subseteq X$ with $A_1 \cup A_2 = A$, one has $A = A_1$ or $A = A_2$.

With this definition, the empty set $\varnothing$ is irreducible; by soberness of the corresponding stably compact space (see Remark 4.2), the non-empty irreducible closed subset $A \subseteq X$ are precisely the subsets of the form $A = {\uparrow} x$, for some $x \in X$.

Proposition 6.7. *Let X be a separated ordered compact space and $A \subseteq X$ a closed upper subset of X. Then the following assertions hold.*

(1) $A \neq \varnothing$ if and only if Φ_A satisfies (Top).
(2) A is irreducible if and only if Φ_A satisfies (Ten).

Proof. (1) is clear, and so is the implication "$\implies$" in (2). Assume now that Φ_A satisfies (Ten) and let $A_1, A_2 \subseteq X$ be closed upper subsets with $A_1 \cup A_2 = A$. Let $x \notin A_1$ and $y \notin A_2$. We find $\psi_1, \psi_2 \in CX$ with

$$\psi_1(x) = 1, \qquad \psi_2(y) = 1, \qquad \forall z \in A \,.\, \psi_1(z) = 0 \text{ or } \psi_2(z) = 0.$$

Therefore

$$0 = \Phi_A(\psi_1 \otimes \psi_2) = \Phi_A(\psi_1) \otimes \Phi_A(\psi_2).$$

By Corollary 3.6, $\Phi_A(\psi_1) = 0$ or, for some $n \in \mathbb{N}$, $\Phi_A(\psi_2^n) = \Phi_A(\psi_2)^n = 0$, hence $x \notin A$ or $y \notin A$. We conclude that $A = A_1$ or $A = A_2$. □

Recall from Example 5.6 that a relation is a function if and only if it is a comonoid in the monoidal category Rel. Hence, we have the following corollary of Proposition 6.7.

Corollary 6.8. *Let $\varphi : X \nrightarrow Y$ in $\mathsf{Ord_sComp}_{\mathbb{V}}$. Then:*

(1) φ is a total relation if and only of $C\varphi$ preserves 1.
(2) φ is a partial function if and only if $C\varphi$ preserves $\otimes$.

Our next goal is to invert the process $A \mapsto \Phi_A$. Firstly, following [Shapiro, 1992], we introduce the subsequent notation.

- For every map $\psi : X \to [0,1]$, $\mathcal{Z}(\psi) = \{x \in X \mid \psi(x) = 0\}$ denotes the zero-set of ψ. If ψ is a monotone and continuous map $\psi : X \to [0,1]^{\mathrm{op}}$, then $\mathcal{Z}(\psi)$ is an closed upper subset of X.
- For every map $\Phi : CX \to [0,1]$, we put
$$\mathcal{Z}(\Phi) = \bigcap \{\mathcal{Z}(\psi) \mid \psi \in CX,\, \Phi(\psi) = 0\} \subseteq X.$$
Note that $\mathcal{Z}(\Phi)$ is a closed upper subset of X.

There is arguably a more natural candidate for an inverse of j_X. First note that, given a set $\{A_i \mid i \in I\}$ of closed upper subsets of X with $A = \overline{\bigcup_{i \in I} A_i}$, for every $\psi \in CX$ one verifies

$$\Phi_A(\psi) = \sup_{x \in \bigcup_{i \in I} A_i} \psi(x) = \sup_{i \in I} \Phi_{A_i}(\psi).$$

Hence, the monotone map j_X preserves infima[2] and therefore has a left adjoint which sends a morphism $\Phi : CX \to [0,1]$ to

$$\mathcal{A}(\Phi) = \bigcap_{\psi \in CX} \psi^{-1}[0, \Phi(\psi)].$$

In the sequel it will be convenient to consider the maps $\mathcal{Z}$ and $\mathcal{A}$ defined on the set $\{\Phi : CX \to [0,1]\}$ of all maps from CX to $[0,1]$. We have the following elementary properties.

Lemma 6.9. *Let X be a separated ordered compact space X.*

(1) The maps $\mathcal{A}, \mathcal{Z} : \{\Phi : CX \to [0,1]\} \to VX$ are monotone.
(2) $\mathcal{A}(\Phi) \subseteq \mathcal{Z}(\Phi)$, for every map $\Phi : CX \to [0,1]$.
(3) For every $A \in VX$, $\mathcal{Z} \cdot j_X(A)) = A = \mathcal{A} \cdot j_X(A))$.
(4) For every map $\Phi : CX \to [0,1]$ and every $\psi \in CX$, $j_X \cdot \mathcal{A}(\Phi)(\psi) \leq \Phi(\psi)$.

Corollary 6.10. *For every separated ordered compact space X, the map $j_X : VX \to GCX$ is an order-embedding.*

Now, we wish to give conditions on $\Phi : CX \to [0,1]$ so that j_X restricts to a bijection between VX and the subset of $\{\Phi : CX \to [0,1]\}$ defined by these conditions. In particular, we consider:

(A) For all $x \in X$ and all $\psi \in CX$, if $\psi(x) > \Phi(\psi) = 0$, then there exists some $\bar{\psi} \in CX$ with $\bar{\psi}(x) = 1$ and $\Phi(\bar{\psi}) = 0$.

Lemma 6.11. *Let X be a separated ordered compact space.*

(1) If $\Phi : CX \to [0,1]$ satisfies (Mon), (Act) and (Ten)$_{lax}$, then Φ satisfies (A).
(2) If the quantale $[0,1]$ does not have nilpotent elements and $\Phi : CX \to [0,1]$ satisfies (Mon) and (Act), then Φ satisfies (A).

Proof. Assume $\psi(x) > \Phi(\psi) = 0$. Put $v = \psi(x)$ and take u with $0 < u < v$. Put $A = \psi^{-1}([0,u])$. By Proposition 4.5, there is some $\bar{\psi} \in CX$ with $A \subseteq \mathcal{Z}(\bar{\psi})$ and $\bar{\psi}(x) = 1$. Furthermore,

$$u \otimes \bar{\psi} \leq u \wedge \bar{\psi} \leq \psi$$

and therefore $u \otimes \Phi(\bar{\psi}) \leq \Phi(\psi) = 0$. Since $u \neq 0$, we get $\Phi(\bar{\psi})^n = 0$ for some $n \in \mathbb{N}$. If there are no nilpotent elements, then $\Phi(\bar{\psi}) = 0$. In general, using condition (Ten)$_{\text{lax}}$ we obtain $\Phi(\psi^n) \leq \Phi(\psi')^n = 0$ and $\psi^n(x) = 1$. ☐

Inspired by Shapiro's proof we get the following result.

Proposition 6.12. *Let X be a separated ordered compact space. For every $\Phi : CX \to [0,1]$ satisfying (Mon), (Act), (Sup) and (A),*

$$\Phi(\psi) \leq j_X \cdot \mathcal{Z}(\Phi)(\psi),$$

for all $\psi \in CX$.

Proof. Let $\psi \in CX$, we wish to show that $\Phi(\psi) \leq \sup_{x \in Z(\Phi)} \psi(x)$. To this end, consider an element $u \in [0,1]$ with $\sup_{x \in Z(\Phi)} \psi(x) < u$. Put

$$U = \{x \in X \mid \psi(x) < u\}.$$

Clearly, U is open and $Z(\Phi) \subseteq U$. Let now $x \in X \setminus Z(\Phi)$. There is some $\psi' \in CX$ with $\Phi(\psi') = 0$ and $\psi'(x) \neq 0$; by (A) we may assume $\psi'(x) = 1$. Let now $\alpha < 1$. For every $\psi' \in CX$ we put

$$\operatorname{supp}_\alpha(\psi') = \{x \in X \mid \psi'(x) > \alpha\}.$$

By the considerations above,

$$X = U \cup \bigcup \{\operatorname{supp}_\alpha(\psi') \mid \psi' \in C(X), \Phi(\psi') = 0\};$$

[2]Note that the order is reversed.

since X is compact, we find $\psi_1, \dots, \psi_n$ with $\Phi(\psi_i) = 0$ and

$$X = U \cup \operatorname{supp}_\alpha(\psi_1) \cup \dots \cup \operatorname{supp}_\alpha(\psi_n).$$

Hence,

$$\alpha \otimes \psi \leq u \vee (\psi_1 \otimes \psi) \vee \dots \vee (\psi_n \otimes \psi),$$

and therefore

$$\alpha \otimes \Phi(\psi) \leq u \vee \Phi(\psi_1 \otimes \psi) \vee \dots \vee \Phi(\psi_n \otimes \psi) \leq u \vee \Phi(\psi_1) \vee \dots \Phi(\psi_n) = u. \qquad \square$$

Hence, under the conditions of the proposition above, we have

$$\sup_{x \in \mathcal{A}(\Phi)} \psi(x) \leq \Phi(\psi) \leq \sup_{x \in \mathcal{Z}(\Phi)} \psi(x),$$

for all $\psi \in CX$. We investigate now conditions on $\Phi : CX \to [0,1]$ which guarantee $\mathcal{Z}(\Phi) = \mathcal{A}(\Phi)$.

Proposition 6.13. *Assume that $\otimes = *$ is the multiplication or $\otimes = \odot$ is the Łukasiewicz tensor. If Φ satisfies (Mon), (Act) and (Ten)$_{lax}$ then $\mathcal{Z}(\Phi) = \mathcal{A}(\Phi)$.*

Proof. We consider first $\otimes = *$, in this case the proof is essentially taken from [Shapiro, 1992]. For every $\psi \in CX$ and every open lower subset $U \subseteq X$ with $U \cap Z(\Phi) \neq \varnothing$, we show that $\inf_{x \in U} \psi(x) \leq \Phi(\psi)$. To see this, put $u = \inf_{x \in U} \psi(x)$. Since there exists $z \in U \cap Z(\Phi)$, there is some $\psi' \in CX$ with $U^{\complement} \subseteq Z(\psi')$ and $\psi'(z) = 1$; thus $\Phi(\psi') \neq 0$. Then $u * \psi' \leq \psi * \psi'$ and therefore $u * \Phi(\psi') \leq \Phi(\psi) * \Phi(\psi')$. Since $\Phi(\psi') \neq 0$, we obtain $u \leq \Phi(\psi)$.

Let $x \in \mathcal{Z}(\Phi)$, $\psi \in CX$ and $v > \Phi(\psi)$. Put $U = \{x \in X \mid \psi(x) > v\}$. By the discussion above, $U \cap \mathcal{Z}(\Phi) = \varnothing$, hence $\psi(x) \leq v$. Therefore we conclude that $x \in \mathcal{A}(\Phi)$.

Consider now $\otimes = \odot$. Let $x \notin \mathcal{A}(\Phi)$. Then, there is some $\psi \in CX$ with $\psi(x) > \Phi(\psi)$. With $u = \psi(x)$, we obtain

$$\hom(u, \psi(x)) = 1 > \hom(u, \Phi(\psi)) = u \pitchfork \Phi(\psi) \geq \Phi(u \pitchfork \psi),$$

using Remark 2.14 and that $\hom(u, -) : [0,1] \to [0,1]$ is monotone and continuous. Therefore we may assume that $\psi(x) = 1$. Since $\Phi(\psi) < 1$, there is some $n \in \mathbb{N}$ with $\Phi(\psi)^n = 0$, hence $\psi^n(x) = 1$ and $\Phi(\psi^n) = 0$. We conclude that $x \notin \mathcal{Z}(\Phi)$. $\square$

From the results above we obtain:

Theorem 6.14. *Assume that $\otimes = *$ is the multiplication or $\otimes = \odot$ is the Łukasiewicz tensor. Then the monad morphism j between the monad $\mathbb{V}$ on $\mathsf{Ord_sComp}$ and the monad induced by the adjunction $C \dashv G$ of Proposition 6.4 is an isomorphism. Therefore the functor*

$$C : \mathsf{Ord_sComp}_{\mathbb{V}} \longrightarrow [0,1]\text{-}\mathsf{LaxGLat}^{\mathrm{op}}$$

is fully faithful.

For $\Phi : CY \to CX$ in $[0,1]$-$\mathsf{LaxGLat}$, the corresponding distributor $\varphi : X \to\!\!\!\!\circ\; Y$ is given by

$$x \,\varphi\, y \iff y \in \bigcap_{\Phi(\psi)(x)=0} Z(\psi).$$

From Corollary 6.8 one obtains:

Corollary 6.15. *Assume that $\otimes = *$ is the multiplication or $\otimes = \odot$ is the Łukasiewicz tensor. Then the functor*

$$C : \mathsf{Ord_sComp} \longrightarrow [0,1]\text{-}\mathsf{GLat}^{\mathrm{op}}$$

is fully faithful.

The following examples show that Theorem 6.14 and Corollary 6.15 cannot be generalised to arbitrary continuous quantale structures on $[0,1]$; not even if, in the case of Theorem 6.14, we restrict $\mathsf{Ord_sComp}_{\mathbb{V}}$ to the full subcategory $\mathsf{CompHaus}_{\mathbb{V}}$. However, in Theorem 6.23 we show that Corollary 6.15 still holds if we restrict $\mathsf{Ord_sComp}$ to the full subcategory $\mathsf{CompHaus}$.

Examples 6.16. Consider $\otimes = \wedge$.

- For $X = 1$, the set $V1$ contains two elements; however, for every $\alpha \in [0,1]$, the map $\Phi = \alpha \wedge - : [0,1] \to [0,1]$ satisfies (Mon), (Act), (Sup) and $(\text{Ten})_{\text{lax}}$.
- For the compact Hausdorff space $X = \{0,1\}$, the set VX contains four elements; however, for every $\alpha \in [0,1]$, the map
$$\Phi_\alpha : [0,1] \times [0,1] \longrightarrow [0,1],\ (u,v) \longmapsto u \vee (\alpha \wedge v)$$
satisfies (Mon), (Act), (Sup) and $(\text{Ten})_{\text{lax}}$ (but, in general, not (Ten)); moreover, $\alpha = \Phi_\alpha(0,1)$ and therefore $\Phi_\alpha \neq \Phi_\beta$ for $\alpha \neq \beta$.
- For the separated ordered compact space $X = \{0 \geq 1\}$, $CX = \{(u,v) \in [0,1] \times [0,1] \mid u \leq v\}$ and VX contains three elements; however, for every $\alpha \in [0,1]$, the map
$$\Phi_\alpha : CX \longrightarrow [0,1],\ (u,v) \longmapsto u \vee (\alpha \wedge v)$$
satisfies (Mon), (Act), (Sup), (Ten) and (Top). In comparison with the previous example, the non-discrete order of X allows to show that Φ_α satifies (Ten). To see this, take $(u,v),(u',v') \in CX$. Then,
$$\begin{aligned}\Phi_\alpha(u,v) \wedge \Phi_\alpha(u',v') &= (u \wedge u') \vee (\alpha \wedge u \wedge v') \vee (\alpha \wedge v \wedge u') \vee (\alpha \wedge v \wedge v')\\ &= (u \wedge u') \vee (\alpha \wedge v \wedge v') = \Phi_\alpha\big((u,v) \wedge (u',v')\big).\end{aligned}$$

To deal with the general case, we introduce the following condition on a map $\Phi : CX \to [0,1]$ where $\ominus$ denotes truncated minus on $[0,1]$.

(Min): For every $u \in [0,1]$ and every $\psi \in CX$, $\Phi(\psi \ominus u) = \Phi(\psi) \ominus u$.

Clearly, for every closed upper subset $A \subseteq X$, the map $\Phi_A : CX \to [0,1]$ satisfies (Min).

Proposition 6.17. *Let X be a separated ordered compact space and $\Phi : CX \to [0,1]$ a map satisfying (Min). Then*
$$\mathcal{A}(\Phi) = \mathcal{Z}(\Phi).$$

Proof. Assume $x \notin \mathcal{A}(\Phi)$. Then there is some $\psi \in CX$ with $\psi(x) > \Phi(\psi)$. Put $u = \Phi(\psi)$. Then $\Phi(\psi \ominus u) = 0$ and $(\psi \ominus u)(x) > 0$, hence $x \notin \mathcal{Z}(\Phi)$. $\square$

Therefore we obtain:

Proposition 6.18. *Let X be a separated ordered compact space. The map*
$$j_X : VX \longrightarrow \{\Phi : CX \to [0,1] \mid \Phi \textit{ satisfies (Mon), (Act), (Sup), } (\text{Ten})_{lax} \textit{ and (Min)}\},\ A \longmapsto \Phi_A$$
is bijective. If the quantale $[0,1]$ does not have nilpotent elements, then j_X is bijective even if the condition $(\text{Ten})_{lax}$ is dropped on the right hand side.

Accordingly, we introduce the categories
$$[0,1]\text{-}\mathsf{GLat}_\ominus \qquad\qquad \text{and} \qquad\qquad [0,1]\text{-}\mathsf{LaxGLat}_\ominus$$
defined as $[0,1]$-GLat and $[0,1]$-$\mathsf{LaxGLat}$ respectively, but the objects have an additional action $\ominus : X \times [0,1] \to X$ and the morphisms preserve it. With the action $\ominus : [0,1] \times [0,1] \to [0,1]$, $(u,v) \mapsto u \ominus v$, the $[0,1]$-category $[0,1]$ is an object of both categories. As before (see Proposition 6.4 and Theorem 6.14), we obtain:

Theorem 6.19. *Under Assumption 6.2, the dualising object $([0,1]^{\mathrm{op}},[0,1])$ induces a natural dual adjunction*
$$\begin{tikzcd} \mathsf{Ord}_{\mathsf{s}}\mathsf{Comp} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] & ([0,1]\text{-}\mathsf{LaxGLat}_\ominus)^{\mathrm{op}}. \arrow[l, bend left, "G"] \end{tikzcd}$$

Here CX is given by $\mathsf{Ord_sComp}(X,[0,1]^{\mathrm{op}})$ with all operations defined pointwise, and GA is the space $[0,1]\text{-}\mathsf{LaxGLat}_{\ominus}(A,[0,1])$ equipped with the initial topology with respect to all evaluation maps

$$\mathrm{ev}_a : [0,1]\text{-}\mathsf{LaxGLat}_{\ominus}(A,[0,1]) \longrightarrow [0,1]^{\mathrm{op}},\ \Phi \longmapsto \Phi(a).$$

Furthermore, we obtain a commutative diagram

$$\begin{tikzcd} \mathsf{Ord_sComp}_{\mathbb{V}} \arrow[rr, "C"] & & ([0,1]\text{-}\mathsf{LaxGLat}_{\ominus})^{\mathrm{op}}, \\ & \mathsf{Ord_sComp} \arrow[ul] \arrow[ur, "C"'] & \end{tikzcd}$$

of functors, and the induced monad morphism j between $\mathbb{V}$ and the monad induced by $C \dashv G$ is an isomorphism. Therefore the functor

$$C : \mathsf{Ord_sComp}_{\mathbb{V}} \longrightarrow ([0,1]\text{-}\mathsf{LaxGLat}_{\ominus})^{\mathrm{op}}$$

is fully faithful, and so is the functor

$$C : \mathsf{Ord_sComp} \longrightarrow ([0,1]\text{-}\mathsf{GLat}_{\ominus})^{\mathrm{op}}.$$

Remark 6.20. Once we know that $C : \mathsf{Ord_sComp} \to ([0,1]\text{-}\mathsf{GLat}_{\ominus})^{\mathrm{op}}$ is fully faithful, we can add on the right hand side further operations if they can be transported pointwise from $[0,1]$ to CX. For instance, if $\hom(u,-) : [0,1] \to [0,1]$ is continuous, then CX has u-powers with $(\psi \pitchfork u)(x) = \hom(u,\psi(x))$, for all $x \in X$. Furthermore, every morphism $\Phi : CX \to CY$ in $[0,1]\text{-}\mathsf{GLat}_{\ominus}$ preserves u-powers.

In 1983, Banaschewski showed that $\mathsf{CompHaus}$ fully embeds into the category of distributive lattices equipped with constants from $[0,1]$ and constant preserving lattice homomorphisms. As we pointed out in Remark 2.7, instead of adding constants to the lattice CX of continuous $[0,1]$-valued functions, one could as well consider an action $u \wedge \psi$ of $[0,1]$ on the lattice CX. Therefore Banaschewski's result should appear as a special case of Theorem 6.14 (for $\otimes = \wedge$). Unfortunately, this is not immediately the case since we need the additional operation $\ominus$. However, using some arguments of [Banaschewski, 1983], we finish this section showing that $\Phi_{Z(\Phi)} = \Phi$, for every compact Hausdorff space and every $\Phi : CX \to [0,1]$ in $[0,1]\text{-}\mathsf{GLat}$.

In analogy to Proposition 6.7, we have:

Proposition 6.21. *Let X be a separated ordered compact space and assume that $\Phi : CX \to [0,1]$ satisfies (Mon), (Act), (Sup)and (Ten)$_{lax}$.*

(1) If Φ satisfies also (Top), then $\mathcal{Z}(\Phi) \neq \varnothing$.
(2) If Φ satisfies also (Ten), then $\mathcal{Z}(\Phi)$ is irreducible (see Definition 6.6).

Proof. To see the first implication: $1 = \Phi(1) \leq \sup_{x \in \mathcal{Z}(\Phi)} 1$, hence $\mathcal{Z}(\Phi) \neq \varnothing$. The proof of the second one is the same as the corresponding proof for Proposition 6.7. ☐

Lemma 6.22. *Let X be a compact Hausdorff space and $\Phi : CX \to [0,1]$ in $[0,1]\text{-}\mathsf{GLat}$. We denote by x_0 the unique element of X with $\mathcal{Z}(\Phi) = \{x_0\}$. Then, for every $\psi \in CX$, $\psi(x_0) = \Phi(\psi)$.*

Proof. By Proposition 6.12, $\Phi(\psi) \leq \psi(x_0)$. To see the reverse inequality, let $u < \psi(x_0)$. Then $x_0 \notin \{x \in X \mid \psi(x) \leq u\}$, therefore there is some $\psi' \in CX$ with $\psi'(x_0) = 0$ and ψ' is constant 1 on $\{x \in X \mid \psi(x) \leq u\}$. Hence, $u \vee \psi' \leq \psi \vee \psi'$. Since $\Phi(\psi') \leq \psi'(x_0) = 0$, we conclude that $u = \Phi(u) \leq \Phi(\psi)$. ☐

Theorem 6.23. *Under Assumption 6.2, the functor*

$$C : \mathsf{CompHaus} \longrightarrow [0,1]\text{-}\mathsf{GLat}^{\mathrm{op}}$$

is fully faithful.

7. A Stone–Weierstrass theorem for $[0,1]$-categories

In this section we adapt the classical Stone–Weierstraß approximation theorem [Stone, 1948a,b] to the context of $[0,1]$-categories, which is an important step towards identifying the image of the fully faithful functor

$$C : \mathsf{Ord_sComp}_{\mathbb{V}} \longrightarrow ([0,1]\text{-}\mathsf{LaxGLat}_{\ominus})^{\mathrm{op}}.$$

To do so, *we continue working under Assumption 6.2.*

We recall that, for every separated ordered compact space X, the $\mathcal{V}$-category CX is finitely cocomplete with $[0,1]$-category structure

$$d(\psi_1,\psi_2) = \inf_{x\in X} \hom(\psi_1(x),\psi_2(x)),$$

for all $\psi_1,\psi_2 \in CX$. Furthermore, by Theorem 2.12, for all $M \subseteq CX$ and $\psi \in CX$ we have $\psi \in \overline{M}$ if and only if, for every $u < 1$, there is some $\psi' \in M$ with $u \le d(\psi,\psi')$ and $u \le d(\psi',\psi)$. Relative to a subset $L \subseteq CX$, we consider the following separation axiom on an separated ordered compact space X:

(Sep): for every $(x,y) \in X \times X$, with $x \not\ge y$, there exists $\psi \in L$ and an open neighbourhood U_y of y such that $\psi(x) = 1$ and, for all $z \in U_y$, $\psi(z) = 0$.

Lemma 7.1. *Let $L \subseteq CX$ be closed in CX under finite suprema, the monoid structure and the action of $[0,1]$; that is, for all $\psi_1,\psi_2 \in L$ and $u \in [0,1]$, $\psi_1 \vee \psi_2 \in L$, $\psi_1 \otimes \psi_2 \in L$, $1 \in L$ and $u \otimes \psi_1 \in L$. Let $\psi \in CX$. If the map $\hom : \mathrm{im}(\psi) \times [0,1] \to [0,1]$ is continuous and L satisfies (Sep), then $\psi \in \overline{L}$.*

Proof. Fix $x \in X$. Let $(\psi_y)_{y\in X}$ be the family of functions defined in the following way.

- If $y \not\le x$, then let ψ_y be a function guaranteed by (Sep) and U_y the corresponding neighborhood.
- If $y \le x$, then ψ_y is the constant function function $\psi(x)$.

By hypothesis, the functions $\hom(\psi(x),-) : [0,1] \to [0,1]$ and ψ are continuous. Thus, the set

$$U_x = \{z \in X \mid u < \hom(\psi(x),\psi(z))\}$$

is an open neighborhood of every $y \le x$, and for such $y \in X$ we put $U_y = U_x$. Consenquently, the collection of sets U_y $(y \in X)$ is an open cover of X. By compactness of X, there exists a finite subcover $U_{y_1},\dots,U_{y_n},U_x$ of X. Considering the corresponding functions $\psi_{y_1},\dots,\psi_{y_n},\psi_x$, we define $\phi_x = \psi_{y_1} \otimes \dots \otimes \psi_{y_n} \otimes \psi_x$.

By construction, ϕ_x has the following properties:

- $\phi_x(x) = \psi(x)$, since $\psi_{y_i}(x) = 1$ for $1 \le i \le n$ and $\psi_x(x) = \psi(x)$;
- for every $z \in X$, $u \otimes \phi_x(z) \le \psi(z)$, since $z \in U_x$ or $z \in U_{y_i}$, for some i.

Now, for every $x \in X$ the set

$$V_x = \{z \in X \mid u < \hom(\psi(z),\phi_x(z))\}$$

is open because the functions $\hom : \mathrm{im}(\psi) \times [0,1] \to [0,1]$, ϕ_x and ψ are continuous. Therefore the collection of the sets V_x is an open cover of X. Again, by compactness of X, there exists a finite subcover $V_{x_1},\dots,V_{x_m}$ of X. By defining $\phi = \phi_{x_1} \vee \dots \vee \phi_{x_m}$ we obtain a function in L such that for every $z \in X$:

- $u \otimes \phi(z) = \displaystyle\bigvee_{j=1}^{m} u \otimes \phi_{x_j}(z) \le \bigvee_{j=1}^{m} u \otimes \psi_{y_i}(z)$, for some ψ_{y_i} such that $u \otimes \psi_{y_i} \le \psi(z)$;
- $u \otimes \psi(z) \le \phi(z)$. ☐

Remark 7.2. For the Łukasiewicz tensor, the lemma above affirms that L is dense in CX in the usual sense since, in this case,

$$\hom(u,v) \ge 1-\varepsilon \iff \max(v-u,0) \le \varepsilon,$$

for all $u,v \in [0,1]$. However, if the tensor product is multiplication, the function $\hom : [0,1]\times[0,1] \to [0,1]$ is not continuous in $(0,0)$; which requires us to add a further condition involving truncated minus (see Lemma 7.4). Finally, if the tensor is the infimum, we cannot expect to obtain a useful Stone–Weierstraß theorem using this closure. For example, for the separated ordered compact space $1 = \{*\}$ the topology in $CX \simeq [0,1]$ is generated by the sets $\{u\}$ and $]u,1]$ with $u \neq 1$. For $x \neq 1$ and $M \subseteq [0,1]$, this means that the seemingly weaker condition $x \in \overline{M}$ implies that $x \in M$.

Lemma 7.3. *Let $\otimes = \odot$ be the Łukasiewicz tensor and $L \subseteq CX$. Assume that L is closed in CX under the monoid structure and u-powers, for all $u \in [0,1]$, and that the cone $(f : X \to [0,1]^{\mathrm{op}})_{f \in L}$ is initial; that is, for all $x, y \in X$, $x \geq y$ if and only if, for all $\psi \in L$, $\psi(x) \leq \psi(y)$. Then L satisfies (Sep).*

Proof. Let $(x,y) \in X \times X$ with $x \not\geq y$. By hyphotesis, there exists $\psi \in L$ and $c \in [0,1]$ such that $\psi(x) > c > \psi(y)$. Let $u = \psi(x)$. Since L is closed for u-powers then $\psi' = \hom(u,\psi) \in L$. By Corollary 3.6 there exists $n \in \mathbb{N}$ such that $c^n = 0$. Therefore $\psi'^n(x) = 1$ and for all $z \in U_y = \psi^{-1}[0,c[$, $\psi'^n(z) = 0$. □

Lemma 7.4. *Let $\otimes = *$ be the multiplication and $L \subseteq CX$. Assume that L is closed in CX under u-powers and $- \ominus u$, for all $u \in [0,1]$, and that the cone $(f : X \to [0,1]^{\mathrm{op}})_{f \in L}$ is initial. Then L satisfies (Sep).*

Proof. Let $(x,y) \in X \times X$ with $x \not\geq y$. By hyphotesis, there exists $\psi \in L$ and $c \in [0,1]$ such that $\psi(x) > c > \psi(y)$. Let $\psi' = \psi \ominus c$ and $u = \psi'(x)$. Let $\psi'' = \hom(u,\psi') \in L$ and $U_y = \psi'^{-1}[0,c[$. Clearly, $\psi''(x) = 1$ and, since $u > 0$, for all $z \in U_y$ we obtain $\psi''(z) = 0$. □

The results above tell us that certain $[0,1]$-subcategories of CX are actually equal to CX if they are closed in CX. To ensure this property, we will work now with Cauchy-complete $[0,1]$-categories. First we have to make sure that the $[0,1]$-category CX is Cauchy-complete.

Lemma 7.5. *The subset*

$$\{(u,v) \mid u \leq v\} \subseteq [0,1] \times [0,1]$$

of the $[0,1]$-category $[0,1] \times [0,1]$ is closed.

Proof. Just observe that $\{(u,v) \mid u \leq v\}$ can be presented as the equaliser of the $[0,1]$-functors $\wedge : [0,1] \times [0,1] \to [0,1]$ and $\pi_1 : [0,1] \times [0,1] \to [0,1]$. □

Corollary 7.6. *For every ordered set X (with underlying set $|X|$), the subset*

$$\mathsf{Ord}(X,[0,1]^{\mathrm{op}}) \subseteq [0,1]^{|X|}$$

of the $[0,1]$-category $[0,1]^{|X|}$ is closed.

With $U : \mathsf{Set} \to \mathsf{Set}$ denoting the ultrafilter functor, we write

$$\xi : U[0,1] \longrightarrow [0,1],\ \xi(\mathfrak{x}) = \sup_{A \in \mathfrak{x}} \inf_{u \in A} u = \inf_{A \in \mathfrak{x}} \sup_{u \in A} u.$$

for the convergence of the Euclidean topology of $[0,1]$.

Lemma 7.7. *For every set X and every ultrafilter $\mathfrak{x}$ on X, the map*

$$\Phi_{\mathfrak{x}} : [0,1]^X \longrightarrow [0,1],\ \psi \longmapsto \xi \cdot U\psi(\mathfrak{x})$$

is a $[0,1]$-functor.

Proof. Since domain and codomain of $\Phi_{\mathfrak{x}}$ are both $\mathcal{V}$-copowered, the assertion follows from

$$\xi \cdot U\psi \leq \xi \cdot U\psi' \qquad \text{and} \qquad \xi \cdot U(\psi \otimes u) = (\xi \cdot U\psi) \otimes u,$$

for all $u \in [0,1]$ and $\psi, \psi' \in [0,1]^X$ with $\psi \leq \psi'$. □

Corollary 7.8. *For every compact Hausdorff space X, the subset*

$$\mathsf{CompHaus}(X,[0,1]) \subseteq [0,1]^{|X|}$$

of the $[0,1]$-category $[0,1]^{|X|}$ is closed.

Proof. For an ultrafilter $\mathfrak{x} \in UX$ with convergence point $x \in X$, a map $\psi : X \to [0,1]$ preserves this convergence if and only if ψ belongs to the equaliser of $\Phi_{\mathfrak{x}}$ and π_x. □

Proposition 7.9. *For every separated ordered compact space X, the $[0,1]$-category CX is Cauchy-complete.*

We will now introduce a category A of $[0,1]$-categories which depends on the chosen tensor $\otimes$ on $[0,1]$.

For the Łukasiewicz tensor $\otimes = \odot$: A is the category with objects all $[0,1]$-powered objects in the category $[0,1]$-GLat, and morphisms all those arrows in $[0,1]$-GLat which preserve powers by elements of $[0,1]$.

For the multiplication $\otimes = *$: A is the category with objects all $[0,1]$-powered objects in the category $[0,1]$-$\mathsf{GLat}_\ominus$, and morphisms all those arrows in $[0,1]$-$\mathsf{GLat}_\ominus$ which preserve powers by elements of $[0,1]$.

Remark 7.10. The category A over Set is a $\aleph_1$-ary quasivariety and, moreover, a full subcategory of a finitary variety. Therefore the isomorphisms in A are precisely the bijective morphisms.

Proposition 7.11. *Assume that $\otimes = *$ is the multiplication or $\otimes = \odot$ is the Łukasiewicz tensor. Let $m : A \to CX$ be an injective morphism in A so that the cone $(m(a) : X \to [0,1]^{\mathrm{op}})_{a \in A}$ is point-separating and initial with respect to the canonical forgetful functor $\mathsf{Ord_sComp} \to \mathsf{Set}$. Then m is an isomorphism in A if and only if A is Cauchy-complete.*

Proof. Clearly, if m is an isomorphism, then A is Cauchy-complete since CX is so. The reverse implication is clear for $\otimes = \odot$ by Lemmas 7.1 and 7.3. Consider now $\otimes = *$ multiplication. Let $\psi \in CX$. Put $\psi' = \frac{1}{2} * \psi + \frac{1}{2}$, then ψ' is monotone and continuous. By Lemmas 7.1 and 7.4, $\psi' \in \mathrm{im}(m)$ and therefore also $\psi = \hom(\frac{1}{2}, \psi' \ominus \frac{1}{2})$ belongs to CX. □

We say that an object A of A ***has enough characters*** whenever the cone $(\varphi : A \to [0,1])_\varphi$ of all morphisms into $[0,1]$ separates the points of A.

Theorem 7.12. *Let A be an object in A. Then $A \simeq CX$ in A for some separated ordered compact space X if and only if A is Cauchy-complete and has enough characters.*

Proof. If $A \simeq CX$ in A, then clearly A is Cauchy-complete and has enough characters. Assume now that A has this properties. Then, by [Lambek and Rattray, 1979, Proposition 2.4], $X = \hom(A,[0,1])$ is a separated ordered compact space with the initial structure relative to all evaluation maps $\mathrm{ev}_a : X \to [0,1]^{\mathrm{op}}$ $(a \in A)$. The map $m : A \to CX$, $a \mapsto ev_a$ is injective since A has enough characters and satisfies the hypothesis of Proposition 7.11, hence m is an isomorphism. □

Finally, Theorem 7.12 allows us to describe the image of the fully faithful functors of Theorem 6.14 and Corollary 6.15, and we end this section presenting duality results for $\mathsf{Ord_sComp}_{\mathbb{V}}$ and $\mathsf{Ord_sComp}$ where the objects on the dual side should be thought of as "metric distributive lattices". To do so, we consider now the following categories.

- $\mathsf{A}_{[0,1],\mathrm{cc}}$ denotes the full subcategory of A defined by the Cauchy-complete objects having enough characters.
- $\mathsf{B}_{[0,1],\mathrm{cc}}$ denotes the category with the same objects as $\mathsf{A}_{[0,1],\mathrm{cc}}$, and the morphisms of $\mathsf{B}_{[0,1],\mathrm{cc}}$ are the finitely cocontinuous $[0,1]$-functors which laxly preserve the multiplication.

Theorem 7.13. *For $\otimes = *$ the multiplication or $\otimes = \odot$ the Łukasiewicz tensor,*

$$\mathsf{Ord_sComp}_{\mathbb{V}} \simeq \mathsf{B}^{\mathrm{op}}_{[0,1],\mathrm{cc}} \qquad \text{and} \qquad \mathsf{Ord_sComp} \simeq \mathsf{A}^{\mathrm{op}}_{[0,1],\mathrm{cc}}.$$

8. Metric compact Hausdorff spaces and metric Vietoris monads

As we pointed already out in Remark 5.8, the constructions leading to dualities for Kleisli categories seem to be more "canonical" if we work with a monad $\mathbb{T} = (T, m, e)$ satisfying $T1 \simeq [0,1]^{\mathrm{op}}$. In [Hofmann, 2014] we introduce a generalisation of the Vietoris monad with this property in the context of "enriched topological spaces", more precisely, models of topological theories as defined in [Hofmann, 2007]. Such a topological theory involves a Set-monad and a quantale together with an Eilenberg–Moore algebra structure on the underlying set of the quantale, subject to further axioms. In this paper we consider only the ultrafilter monad $\mathbb{U} = (U, m, e)$ and a quantale with underlying lattice $[0,1]$. The convergence of the Euclidean compact Hausdorff topology on $[0,1]$ defines an Eilenberg–Moore algebra structure for the ultrafilter monad:

$$\xi : U[0,1] \longrightarrow [0,1],\ \xi(\mathfrak{x}) = \sup_{A\in\mathfrak{x}} \inf_{u\in A} u = \inf_{A\in\mathfrak{x}} \sup_{u\in A} u.$$

We continue assuming that the multiplication of the quantale $[0,1]$ is continuous and has 1 as neutral element; that is, we continue working under Assumption 6.2. Under these conditions, $\mathcal{U} = (\mathbb{U}, [0,1], \xi)$ is a strict topological theory as defined in [Hofmann, 2007]. In order to keep the amount of background theory small, we do not enter here into the details of monad-quantale enriched categories but give only the details needed to understand the Kleisli category of the $[0,1]$-enriched Vietoris monad. We refer to [Hofmann, 2014, Section 1] for an overview, and a comprehensive presentation of this theory can be found in [Hofmann *et al.*, 2014].

The functor $U : \mathsf{Set} \to \mathsf{Set}$ extends to a 2-functor $U_\xi : [0,1]\text{-}\mathsf{Rel} \to [0,1]\text{-}\mathsf{Rel}$ where $U_\xi X = UX$ for every set X and

$$U_\xi r(\mathfrak{x}, \mathfrak{y}) = \{\xi \cdot Ur(\mathfrak{w}) \mid \mathfrak{w} \in U(X\times Y), U\pi_1(\mathfrak{w}) = \mathfrak{x}, U\pi_2(\mathfrak{w}) = \mathfrak{y}\} = \sup_{A\in\mathfrak{x}, B\in\mathfrak{y}} \inf_{x\in A, y\in B} r(x,y)$$

for all $r : X \to\!\!\!\!\!\to Y$ in $[0,1]$-Rel, $\mathfrak{x} \in UX$ and $\mathfrak{y} \in UY$.

Definition 8.1. A $\mathcal{U}$-***category*** is a set X equipped with a $[0,1]$-relation

$$a : UX \times X \longrightarrow [0,1],$$

subject to the axioms

$$1 = a(e_X(x), x) \qquad \text{and} \qquad U_\xi a(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \le a(m_X(\mathfrak{X}), x),$$

for all $\mathfrak{X} \in UUX$, $\mathfrak{x} \in UX$ and $x \in X$. Given $\mathcal{U}$-categories (X, a) and (Y, b), a $\mathcal{U}$-***functor*** $f : (X,a) \to (Y, b)$ is a map $f : X \to Y$ such that

$$a(\mathfrak{x}, x) \le b(Uf(\mathfrak{x}), f(x)),$$

for all $\mathfrak{x} \in UX$ and $x \in X$.

Clearly, the composite of $\mathcal{U}$-functors is a $\mathcal{U}$-functor, and so is the identity map $1_X : (X,a) \to (X,a)$, for every $\mathcal{U}$-category (X, a). We denote the category of $\mathcal{U}$-categories and $\mathcal{U}$-functors by $\mathcal{U}$-Cat. The most notable example is certainly the case of $\otimes = *$ being multiplication: since $[0,1] \simeq [0,\infty]$, $\mathcal{U}$-Cat is isomorphic to the category App of approach spaces and non-expansive maps (see [Lowen, 1997; Clementino and Hofmann, 2003]).

The category $\mathcal{U}$-Cat comes with a canonical forgetful functor $\mathcal{U}$-$\mathsf{Cat} \to \mathsf{Set}$ which is topological, hence $\mathcal{U}$-Cat is complete and cocomplete and the forgetful functor $\mathcal{U}$-$\mathsf{Cat} \to \mathsf{Set}$ preserves limits and colimits. For example, the product of $\mathcal{U}$-categories (X, a) and (Y, b) can be constructed by first taking the Cartesian product $X \times Y$ of the sets X and Y, and then equipping $X \times Y$ with the structure c defined by

$$c(\mathfrak{w}, (x, y)) = a(U\pi_1(\mathfrak{w}), x) \wedge b(U\pi_2(\mathfrak{w}), y),$$

for all $\mathfrak{w} \in U(X \times Y)$, $x \in X$ and $y \in Y$. More important to us is, however, a different structure c on $X \times Y$ derived from the tensor product of $[0,1]$, namely

$$c(\mathfrak{w}, (x, y)) = a(U\pi_1(\mathfrak{w}), x) \otimes b(U\pi_2(\mathfrak{w}), y).$$

We denote this $\mathcal{U}$-category as $(X,a)\otimes(Y,b)$; in fact, this construction extends naturally to morphisms and yields a functor $-\otimes- : \mathcal{U}\text{-}\mathsf{Cat}\times\mathcal{U}\text{-}\mathsf{Cat}\to\mathcal{U}\text{-}\mathsf{Cat}$.

Every $\mathcal{U}$-category (X,a) defines a topology on the set X with convergence

$$\mathfrak{x}\to x \iff 1=a(\mathfrak{x},x),$$

and this construction defines a functor $\mathcal{U}\text{-}\mathsf{Cat}\to\mathsf{Top}$ which commutes with the forgetful functors to Set. The functor $\mathcal{U}\text{-}\mathsf{Cat}\to\mathsf{Top}$ has a left adjoint $\mathsf{Top}\to\mathcal{U}\text{-}\mathsf{Cat}$ which sends a topological space to the $\mathcal{U}$-category with the same underlying set, say X, and with the discrete convergence

$$UX\times X\longrightarrow[0,1],\;(\mathfrak{x},x)\longmapsto\begin{cases}1 & \text{if } \mathfrak{x}\to x,\\ 0 & \text{otherwise.}\end{cases}$$

This functor allows us to interpret topological spaces as $\mathcal{U}$-categories. Note that, for a topological space X and a $\mathcal{U}$-category Y, we have $X\otimes Y=X\times Y$ in $\mathcal{U}\text{-}\mathsf{Cat}$. There is also a faithful functor $\mathcal{U}\text{-}\mathsf{Cat}\to[0,1]\text{-}\mathsf{Cat}$ which commutes with the forgetful functors to Set and sends a $\mathcal{U}$-category (X,a) to the $[0,1]$-category (X,a_0) where $a_0=a\cdot e_X$; hence $a_0(x,y)=a(e_X(x),y)$, for all $x,y\in X$. We also note that the natural order of the underlying topology of an $\mathcal{U}$-category (X,a) coincides with the order induced by the $[0,1]$-category (X,a_0). We refer to this order as the underlying order of (X,a).

Remark 8.2. An important example of a $\mathcal{U}$-category is given by $[0,1]$ with convergence $(\mathfrak{x},u)\mapsto\hom(\xi(\mathfrak{x}),u)$. In the underlying topology of $[0,1]$, $\mathfrak{x}\to u$ precisely when $\xi(\mathfrak{x})\le u$; hence, the closed subsets are precisely the intervals $[v,1]$ where $v\in[0,1]$. From now on $[0,1]$ refers to this $\mathcal{U}$-category; to distinguish, $[0,1]_e$ denotes the standard compact Hausdorff space with convergence ξ, and $[0,1]_o$ denotes the ordered compact Hausdorff space with the usual order and the Euclidean topology.

The enriched Vietoris monad $\mathbb{V}=(V,m,e)$ on $\mathcal{U}\text{-}\mathsf{Cat}$ sends a $\mathcal{U}$-category X to the $\mathcal{U}$-category VX with underlying set

$$\{\varphi:X\longrightarrow[0,1]\mid\varphi\text{ is a }\mathcal{U}\text{-functor}\},$$

the underlying $[0,1]$-category structure of the $\mathcal{U}$-category VX is given by

$$[\varphi,\varphi']=\inf_{x\in X}\hom(\varphi'(x),\varphi(x)),$$

and therefore

$$\varphi\le\varphi'\iff\varphi(x)\ge\varphi'(x),\;\text{for all } x\in X$$

in its underlying order. It is shown in [Hofmann, 2014] that this monad restricts to the $[0,1]$-enriched counterpart of the category of stably compact spaces and proper maps: the category of separated representable $\mathcal{U}$-categories. A $\mathcal{U}$-category (X,a) is ***representable*** whenever $a\cdot U_\xi a=a\cdot m_X$ and there is a map $\alpha:UX\to X$ with $a=a_0\cdot\alpha$. If (X,a) is separated, then $\alpha:UX\to X$ is unique and an $\mathbb{U}$-algebra structure on X; that is, the convergence of a compact Hausdorff topology on X. The separated representable $\mathcal{U}$-categories are the objects of the category $\mathcal{U}\text{-}\mathsf{Rep}$, a morphism $f:(X,a)\to(Y,b)$ in $\mathcal{U}\text{-}\mathsf{Rep}$ is a $[0,1]$-functor $f:(X,a_0)\to(Y,b_0)$ where f is also continuous with respect to the corresponding compact Hausdorff topologies; that is, $f\cdot\alpha=\beta\cdot Uf$. We also note that the category $\mathcal{U}\text{-}\mathsf{Rep}$ is complete and the inclusion functor $\mathcal{U}\text{-}\mathsf{Rep}\to\mathcal{U}\text{-}\mathsf{Cat}$ preserves limits. For (X,a) in $\mathcal{U}\text{-}\mathsf{Rep}$ with $a=a_0\cdot\alpha$, also $(X,a_0^\circ\cdot\alpha)$ is a separated representable $\mathcal{U}$-category, called the ***dual*** of (X,a) and denoted as $(X,a)^{\mathrm{op}}$. In fact, this construction defines a functor $(-)^{\mathrm{op}}:\mathcal{U}\text{-}\mathsf{Rep}\to\mathcal{U}\text{-}\mathsf{Rep}$ leaving maps unchanged. The $\mathcal{U}$-category $[0,1]$ is separated and representable, with $\hom:\mathcal{V}\nrightarrow\mathcal{V}$ being the underlying $[0,1]$-category structure and $\xi:U[0,1]\to[0,1]$ the convergence of the corresponding compact Hausdorff topology. We note that now $V1$ is isomorphic to $[0,1]^{\mathrm{op}}$. Below we collect some important properties.

Proposition 8.3. *The following assertions hold.*

(1) The maps

$$\wedge : [0,1] \times [0,1] \longrightarrow [0,1], \qquad \vee : [0,1] \times [0,1] \longrightarrow [0,1], \qquad \otimes : [0,1] \otimes [0,1] \longrightarrow [0,1]$$

are morphisms in $\mathcal{U}$-Rep*.*

(2) For every $u \in [0,1]$*,* $u \otimes - : [0,1] \to [0,1]$ *is a morphism of* $\mathcal{U}$-Rep*. Therefore the map* $u \otimes -$ *preserves non-empty infima.*

(3) Let $u \in [0,1]$ *so that* $\hom(u,-)$ *is continuous of type* $\hom(u,-) : [0,1]_e \to [0,1]_e$*. Then* $\hom(u,-) : [0,1] \to [0,1]$ *is a morphism in* $\mathcal{U}$-Rep*.*

(4) $\inf : [0,1]^I \to [0,1]$ *is a* $\mathcal{U}$*-functor, for every set* I*.*

(5) For every $v \in [0,1]$*,* $\hom(-,v) : [0,1]^{\mathrm{op}} \to [0,1]$ *is a* $\mathcal{U}$*-functor.*

Proof. The first assertion follows directly from our Assumption 6.2. Note that $[0,1] \otimes [0,1]$ is in $\mathcal{U}$-Rep, see [Hofmann, 2013, Remark 4.9]. The second assertion is a direct consequence of the first one, and the third one follows form $\hom(u,-) : [0,1] \to [0,1]$ being a $[0,1]$-functor. Regarding the forth assertion, see [Hofmann, 2007, Corollary 5.3]. Regarding the last assertion, the condition of [Hofmann, 2007, Lemma 5.1] can be verified using [Hofmann, 2007, Lemma 3.2]. □

Remark 8.4. Similarly to the connection between stably compact spaces and separated ordered compact spaces (see Remark 4.2), representable $\mathcal{U}$-categories can be seen as compact Hausdorff spaces with a compatible $[0,1]$-category structure. More precisely, in [Tholen, 2009] it is shown that the Set-monad $\mathbb{U}$ extends to a monad on $[0,1]$-Cat, and there is a natural comparison functor $K : ([0,1]\text{-}\mathsf{Cat})^{\mathbb{U}} \to \mathcal{U}\text{-}\mathsf{Cat}$ sending a $[0,1]$-category (X, a_0) with Eilenberg–Moore algebra structure $\alpha : U(X, a_0) = (UX, U_\xi a_0) \to (X, a_0)$ to the $\mathcal{U}$-category $(X, a_0 \cdot \alpha)$. The functor K restricts to an equivalence between the full subcategory of $([0,1]\text{-}\mathsf{Cat})^{\mathbb{U}}$ defined by all separated $[0,1]$-categories and the category $\mathcal{U}$-Rep (see also [Hofmann, 2014] for details).

We do not need to say much about the enriched Vietoris monad $\mathbb{V} = (V, m, e)$ here, it is enough to have a better understanding of the Kleisli category $\mathcal{U}\text{-}\mathsf{Rep}_{\mathbb{V}}$. A morphism $X \to VY$ in $\mathcal{U}$-Rep corresponds to a $[0,1]$-distributor between the underlying $[0,1]$-categories, we call such a distributor a ***continuous*** $[0,1]$***-distributor*** between the separated representable $\mathcal{U}$-categories X and Y. Similar to the classical case, composition in $\mathcal{U}\text{-}\mathsf{Rep}_{\mathbb{V}}$ corresponds to composition of $[0,1]$-relations, and the identity morphism on X is given by a_0 (see [Hofmann, 2014, Section 8]).

The adjunction

$$\begin{tikzcd} \mathcal{U}\text{-}\mathsf{Cat} \arrow[r, bend left, "\text{forgetful}"] \arrow[r, phantom, "\top"] & \mathsf{Top} \arrow[l, bend left, "\text{discrete}"] \end{tikzcd}$$

restricts to an adjunction

$$\begin{tikzcd} \mathcal{U}\text{-}\mathsf{Rep} \arrow[r, bend left, "\text{forgetful}"] \arrow[r, phantom, "\top"] & \mathsf{Ord_sComp}, \arrow[l, bend left, "\text{discrete}"] \end{tikzcd}$$

which allows us to transfer $\mathbb{V}$ to a monad $\mathbb{V} = (V, m, e)$ on $\mathsf{Ord_sComp}$. The Kleisli category $\mathsf{Ord_sComp}_{\mathbb{V}}$ for this monad can be identified with the full subcategory of $\mathcal{U}\text{-}\mathsf{Rep}_{\mathbb{V}}$ defined by the $\mathcal{U}$-categories in the image of $\mathsf{Ord_sComp} \xrightarrow{\text{discrete}} \mathcal{U}\text{-}\mathsf{Rep}$. Furthermore, the Kleisli category of the Vietoris monad of Section 4 on $\mathsf{Ord_sComp}$ can be identified with the subcategory of this "new" Kleisli category $\mathsf{Ord_sComp}_{\mathbb{V}}$ defined by those $[0,1]$-distributors $\varphi : X \mathrel{\circ\!\!\to} Y$ where the map $\varphi : X \times Y \to [0,1]$ takes only values in $\{0,1\}$; we refer to these distributors as 2***-distributors***.

The functor $\mathcal{U}\text{-}\mathsf{Rep} \xrightarrow{\text{forgetful}} \mathsf{Ord_sComp}$ sends the $\mathcal{U}$-category $[0,1]$ to the ordered compact Hausdorff space $[0,1]_o$ and $[0,1]^{\mathrm{op}}$ to $[0,1]_o^{\mathrm{op}}$, the latter follows from the fact that this functor commutes with dualisation. Finally, there is also a canonical adjunction

$$\begin{tikzcd} \mathcal{U}\text{-}\mathsf{Rep} \arrow[r, bend left, "\text{forgetful}"] \arrow[r, phantom, "\top"] & \mathsf{CompHaus} \arrow[l, bend left, "\text{discrete}"] \end{tikzcd}$$

sending a separated representable $\mathcal{U}$-category to its corresponding compact Hausdorff space (see also [Hofmann, 2014, Remark 2.6]).

Since the construction of dual adjunctions typically involves initial lifts, below we give a description of initial cones in $\mathcal{U}$-Rep.

Proposition 8.5. *Let $(\psi_i : (X, a) \to (X_i, a_i))_{i\in I}$ be a point-separating cone in $\mathcal{U}$-Rep. Then the following assertions are equivalent.*

(i) For all $x, y \in X$, $a_0(x, y) = \inf_{i\in I} a_{i0}(\psi_i(x), \psi_i(y))$.
(ii) The cone $(\psi : X \to X_i)_{i\in I}$ is initial with respect to the forgetful functor $\mathcal{U}$-Rep $\to$ CompHaus.
(iii) The cone $(\psi : X \to X_i)_{i\in I}$ is initial with respect to the forgetful functor $\mathcal{U}$-Rep $\to$ Set.
(iv) The cone $(\psi : X \to X_i)_{i\in I}$ is initial with respect to the forgetful functor $\mathcal{U}$-Cat $\to$ Set.

Proof. This follows from the description of initial structures for the functor $([0,1]\text{-}\mathsf{Cat})^{\mathbb{U}} \to [0,1]\text{-}\mathsf{Cat}$ given in [Tholen, 2009, Proposition 3], the fact that every point-separating cone is initial with respect to CompHaus $\to$ Set, and the description of initial structures for the functor $\mathcal{U}$-Cat $\to$ Set in [Hofmann *et al.*, 2014, Proposition III.3.1.1]. □

Definition 8.6. A point-separating cone in $\mathcal{U}$-Rep is called ***initial*** whenever it satisfies the first and hence all of the conditions of Proposition 8.5.

9. Duality theory for continuous enriched distributors

In this section we will use the setting described in Section 8 and aim for results similar to the ones obtained in Section 6 for ordered compact Hausdorff spaces. To do so, *we continue working under Assumption 6.2*. We stress that in the setting of $[0,1]$-distributors the results seem more canonical since now we can dispense the monoid structure and work with $[0,1]$-FinSup instead of $[0,1]$-GLat.

By Propositions 8.3 and 5.4, the dualising object $([0,1]^{\mathrm{op}}, [0,1])$ induces a natural dual adjunction

$$\begin{tikzcd} \mathcal{U}\text{-}\mathsf{Rep} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] & {[0,1]}\text{-}\mathsf{FinSup}^{\mathrm{op}}; \arrow[l, bend left, "G"] \end{tikzcd}$$

here CX has as underlying set all morphisms $X \to [0,1]^{\mathrm{op}}$ in $\mathcal{U}$-Rep. For every separated representable $\mathcal{U}$-category X, the map

$$\hom(X, [0,1]^{\mathrm{op}}) \longrightarrow \hom(VX, [0,1]^{\mathrm{op}}),\ \psi \longmapsto (\varphi \mapsto \psi \cdot \varphi = \sup_{x\in X} \psi(x) \otimes \varphi(x))$$

is certainly a morphism $CX \to CVX$ in $[0,1]$-FinSup, by Theorem 5.7 and Remark 5.8 we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{U}\text{-}\mathsf{Rep}_{\mathbb{V}} \arrow[rr, "C"] & & {[0,1]}\text{-}\mathsf{FinSup}^{\mathrm{op}} \\ & \mathcal{U}\text{-}\mathsf{Rep} \arrow[ul] \arrow[ur, "C"'] & \end{tikzcd}$$

of functors. For $\varphi : X \nrightarrow Y$ in $\mathcal{U}$-$\mathsf{Rep}_{\mathbb{V}}$, we have

$$C\varphi : CY \longrightarrow CX,\ \psi \longmapsto \psi \cdot \varphi.$$

If φ is a 2-distributor, then $C\varphi$ coincides with what was defined in the Section 6.

Remark 9.1. If X is a separated ordered compact space, then, for all $\psi_1,\psi_2\in CX$, the composite

$$X \xrightarrow{\Delta_X} X\times X \simeq X\otimes X \xrightarrow{\psi_1\otimes\psi_2} [0,1]^{\mathrm{op}}\otimes[0,1]^{\mathrm{op}} \simeq ([0,1]\otimes[0,1])^{\mathrm{op}} \xrightarrow{\otimes^{\mathrm{op}}} [0,1]^{\mathrm{op}}$$

is also in $\mathcal{U}$-Rep. Therefore we can still consider $\psi_1\otimes\psi_2\in CX$; however, $C\varphi$ does not need to preserve this operation, not even laxly.

The functor $C:\mathcal{U}\text{-}\mathsf{Rep}_{\mathbb{V}}\to[0,1]\text{-}\mathsf{FinSup}^{\mathrm{op}}$ induces a monad morphism j whose component at X is given by the maps

$$j_X: VX \longrightarrow [0,1]\text{-}\mathsf{FinSup}(CX,[0,1]),\ (\varphi:1 \,{\circ\!\!\!\rightarrow}\, X)\longmapsto \left(\psi\mapsto \psi\cdot\varphi=\sup_{x\in X}(\psi(x)\otimes\varphi(x))\right). \tag{9.i}$$

For every $\Phi: CX\to[0,1]$ in $[0,1]$-FinSup, we define a map $\varphi: X\to[0,1]$ by

$$\varphi(x)=\inf_{\psi\in CX}\hom(\psi(x),\Phi(\psi)).$$

For every $\psi\in CX$,

$$X\xrightarrow{\psi}[0,1]^{\mathrm{op}}\xrightarrow{\hom(-,\Phi(\psi))}[0,1]$$

is a $\mathcal{U}$-functor, and so is $\varphi: X\to[0,1]$ since it can be written as the composite (with $I=CX$)

$$X\longrightarrow[0,1]^I\xrightarrow{\inf}[0,1].$$

In other words, $\varphi\in VX$. For $\Phi,\Phi': CX\to[0,1]$ in $[0,1]$-FinSup with corresponding map $\varphi,\varphi': X\to[0,1]$,

$$\begin{aligned}
[\varphi',\varphi] &= \inf_{x\in X}\hom(\inf_{\overline{\psi}\in CX}\hom(\overline{\psi}(x),\Phi(\overline{\psi})),\inf_{\psi\in CX}\hom(\psi(x),\Phi'(\psi)))\\
&= \inf_{x\in X}\inf_{\psi\in CX}\hom(\inf_{\overline{\psi}\in CX}\hom(\overline{\psi}(x),\Phi(\overline{\psi})),\hom(\psi(x),\Phi'(\psi)))\\
&\geq \inf_{x\in X}\inf_{\psi\in CX}\hom(\hom(\psi(x),\Phi(\psi)),\hom(\psi(x),\Phi'(\psi)))\\
&\geq \inf_{x\in X}\inf_{\psi\in CX}\hom(\Phi(\psi),\Phi'(\psi)) && (\hom(\psi(x),-)\text{ is a }[0,1]\text{-functor})\\
&= [\Phi',\Phi].
\end{aligned}$$

Therefore $\Phi\mapsto\varphi$ defines a $[0,1]$-functor $GCX\to VX$. Furthermore, one easily verifies that these constructions define an adjunction in $[0,1]$-Cat:

Proposition 9.2. *Let X be a separated representable $\mathcal{U}$-category. Then the following assertions hold.*

1. *For every $\varphi\in VX$ and every $x\in X$,*

$$\varphi(x)\leq\inf_{\psi\in CX}\hom(\psi(x),\sup_{y\in X}\psi(y)\otimes\varphi(y)).$$

2. *For every $\Phi: CX\to[0,1]$ in $[0,1]$-FinSup and every $\psi\in CX$,*

$$\sup_{x\in X}\psi(x)\otimes\inf_{\psi'\in CX}\hom(\psi'(x),\Phi(\psi'))\leq\Phi(\psi).$$

Recall from Proposition 4.6 that $[0,1]^{\mathrm{op}}_o$ is an initial cogenerator in $\mathsf{Ord}_{\mathrm{s}}\mathsf{Comp}$; so far we do not know if $[0,1]^{\mathrm{op}}$ is an initial cogenerator in $\mathcal{U}$-Rep. Therefore we define:

Definition 9.3. A separated representable $\mathcal{U}$-category X is called $[0,1]^{\mathrm{op}}$***-cogenerated*** if the cone $(\psi: X\to[0,1]^{\mathrm{op}})_{\psi\in CX}$ is point-separating and initial.

Of course, $[0,1]^{\mathrm{op}}$ is $[0,1]^{\mathrm{op}}$-cogenerated, and so is every separated ordered compact space. Our next question is whether $V:\mathcal{U}\text{-}\mathsf{Rep}\to\mathcal{U}\text{-}\mathsf{Rep}$ restricts to $[0,1]^{\mathrm{op}}$-cogenerated representable $\mathcal{U}$-categories.

Lemma 9.4. *Assume that the tensor product $\otimes$ on $[0,1]$ is either $*$, $\odot$ or $\wedge$. Let X be $[0,1]^{\mathrm{op}}$-cogenerated. Then, for all $x,y\in X$,*

$$a_0(y,x)=\inf_{\psi\in CX,\psi(x)=1}\psi(y).$$

Proof. Since X is $[0,1]^{\mathrm{op}}$-cogenerated, we certainly have

$$a_0(y,x)=\inf_{\psi\in CX}\hom(\psi(x),\psi(y))\le\inf_{\psi\in CX,\psi(x)=1}\psi(y).$$

Assume first that $\otimes=*$ or $\otimes=\odot$. For $\psi\in CX$, put $u=\psi(x)$. Then

$$\hom(\psi(x),\psi(y))=(u\pitchfork\psi)(y)$$

and $(u\pitchfork\psi)(x)=\hom(u,\psi(x))=1$. Since $u\pitchfork\psi\in CX$, the assertion follows.

Assume now that $\otimes=\wedge$. The assertion follows immediately if $a_0(y,x)=1$. Let now $\psi\in CX$, we may assume that $\psi(x)>\psi(y)$. Let $b\in[0,1]$ with $\psi(y)\le b<\psi(x)$. Consider the piecewise linear map $h:[0,1]\to[0,1]$ with $h(v)=v$ for all $v\le b$ and $h(u)=1$ for all $u\ge\psi(x)$. Then $h^{\mathrm{op}}\cdot\psi\in CX$ since $h:[0,1]_e\to[0,1]_e$ is continuous and $h:[0,1]\to[0,1]$ is a $[0,1]$-functor. To see the latter, let $u,v\in[0,1]$. If $\hom(u,v)=1$, then $\hom(h(u),h(v))=1$ since h is monotone. Assume now $u>v$. We distinguish the following cases.

If $\psi(x)\le v$: In this case, $\hom(h(u),h(v))=1\ge\hom(u,v)$.

If $v<\psi(x)\le u$: Now we have $\hom(u,v)=v\le h(v)=\hom(h(u),h(v))$.

If $v<u<\psi(x)$: Assume first that $b\le u$. Then $\hom(u,v)=v$ and $v\le\hom(h(u),h(v))$ since $v\otimes h(u)\le v\le h(v)$. Finally, if $v<u<b$, then the assertion follows from $u=h(u)$ and $v=h(v)$.

We conclude that $h(\psi(y))=\psi(y)$ and $h(\psi(x))=1$. □

Also note that, for every $\psi:X\to[0,1]^{\mathrm{op}}$ in $\mathcal{U}$-Rep, the canonical extension $\psi^{\Diamond}:VX\to[0,1]^{\mathrm{op}}$ of ψ to the free $\mathbb{V}$-algebra VX over X sends $\varphi\in VX$ to $\sup_{x\in X}\varphi(x)\otimes\psi(x)$; and the diagram

$$\begin{tikzcd} VX \arrow[r, "j_X"] \arrow[dr, "\psi^{\Diamond}"'] & GC(X) \arrow[d, "\mathrm{ev}_\psi"] \\ & {[0,1]^{\mathrm{op}}} \end{tikzcd}$$

commutes.

Lemma 9.5. *Assume that the tensor product $\otimes$ on $[0,1]$ is either $*$, $\odot$ or $\wedge$. For every $[0,1]^{\mathrm{op}}$-cogenerated X in $\mathcal{U}$-Rep, the cone $(\psi^{\Diamond}:VX\to[0,1]^{\mathrm{op}})_{\psi\in CX}$ is point-separating.*

Proof. Let $\varphi_1,\varphi_2\in VX$ and $x\in X$ with $\varphi_1(x)<u<\varphi_2(x)$. Let $y\in X$. By Lemma 9.4, and since every $v\otimes-:[0,1]\to[0,1]$ preserves non-empty infima,

$$\inf_{\psi\in CX,\psi(x)=1}(\varphi_1(y)\otimes\psi(y))=\varphi_1(y)\otimes\left(\inf_{\psi\in CX,\psi(x)=1}\psi(y)\right)=\varphi_1(y)\otimes a_0(y,x)\le\varphi_1(x)<u.$$

Therefore there is some $\psi_y\in CX$ with $\psi_y(x)=1$ and $\varphi_1(y)\otimes\psi_y(y)<u$. The composite

$$(X,\alpha)\xrightarrow{\langle\varphi_1,\psi_y\rangle}[0,1]\times[0,1]_e\simeq[0,1]\otimes[0,1]_e\longrightarrow[0,1]\otimes[0,1]\xrightarrow{\otimes}[0,1]$$

is in $\mathcal{U}$-Cat and therefore also continuous with respect to the underlying topologies, which tells us that the set

$$V_y=\{z\in X\mid\varphi_1(z)\otimes\psi_y(z)<u\}$$

is open in the compact Hausdorff space (X,α).

By construction, $(V_y)_{y\in X}$ is an open cover of the compact Hausdorff space (X,α); therefore we find $n\in\mathbb{N}$ and $y_1,\dots,y_n$ in X with $X=V_{y_1}\cup\dots\cup V_{y_n}$. Put $\psi=\psi_{y_1}\wedge\dots\wedge\psi_{y_n}$, clearly, $\psi\in CX$. Then, for all $y\in Y$,

$$\varphi_1(y)\otimes\psi(y)<u\qquad\text{and}\qquad\psi(x)=1;$$

consequently,

$$\sup_{y\in X}(\psi(y)\otimes\varphi_1(y))\le u\qquad\text{and}\qquad\sup_{y\in X}(\psi(y)\otimes\varphi_2(y))\ge\psi(x)\otimes\varphi_2(x)>u,$$

and the assertion follows. □

Corollary 9.6. *Assume that the tensor product $\otimes$ on $[0,1]$ is either $*$, $\odot$ or $\wedge$. For every $[0,1]^{\mathrm{op}}$-cogenerated separated representable $\mathcal{U}$-category X, $j_X : VX \to GCX$ is injective.*

Corollary 9.7. *Assume that the tensor product $\otimes$ on $[0,1]$ is either $*$, $\odot$ or $\wedge$. Let X be a $[0,1]^{\mathrm{op}}$-cogenerated separated representable $\mathcal{U}$-category. Then $j_X : VX \to GCX$ is an embedding in $[0,1]$-Cat and therefore also in $\mathcal{U}$-Rep. Consequently, the cone $(\psi^{\Diamond} : VX \to [0,1]^{\mathrm{op}})_{\psi \in CX}$ is point-separating and initial; and therefore VX is $[0,1]^{\mathrm{op}}$-cogenerated.*

Proof. Since j_X is injective by Corollary 9.6, the inequality in (1) of Proposition 9.2 is actually an equality; and therefore j_X is a split mono in $[0,1]$-Cat. □

We write $\mathcal{U}$-CogRep and $\mathcal{U}$-$\mathsf{CogRep}_{\mathbb{V}}$ to denote the full subcategory of $\mathcal{U}$-Rep respectively $\mathcal{U}$-$\mathsf{Rep}_{\mathbb{V}}$ defined by the $[0,1]^{\mathrm{op}}$-cogenerated separated representable $\mathcal{U}$-categories. Under the conditions of Corollary 9.7, the monad $\mathbb{V}$ can be restricted to $\mathcal{U}$-CogRep and then $\mathcal{U}$-$\mathsf{CogRep}_{\mathbb{V}}$ is indeed the Kleisli category for this monad on $\mathcal{U}$-CogRep.

In the remainder of this section, our arguments use the continuity of the map $\hom(u,-) : [0,1]_e \to [0,1]_e$, for all $u \in [0,1]$. Unfortunately, this property excludes $\otimes = \wedge$; which leaves us with only two choices for $\otimes$:

- $u \otimes v = u * v$ is the multiplication, here $\hom(u,v) = v \oslash u$ is truncated division where $0 \oslash 0 = 1$; and
- $u \otimes v = u \odot v = \max(0, u+v-1)$ is the Łukasiewicz tensor, here $\hom(u,v) = 1 - \max(0, u-v)$.

Lemma 9.8. *Assume that $\otimes = *$ is the multiplication or $\otimes = \odot$ is the Łukasiewicz tensor. Then, for every separated representable $\mathcal{U}$-category X, every $\Phi : CX \to [0,1]$ in $[0,1]$-FinSup and every $x \in X$,*

$$\inf_{\psi \in CX} \hom(\psi(x), \Phi(\psi)) = \inf_{\psi \in CX, \psi(x)=1} \Phi(\psi).$$

Proof. Clearly, the left-hand side is smaller or equal to the right-hand side. Let now $\psi \in CX$ and put $u = \psi(x)$. Then $(\psi \pitchfork u)(x) = \hom(u, \psi(x)) = 1$ and

$$\Phi(\psi \pitchfork u) \le \hom(u, \Phi(\psi)) = \hom(\psi(x), \Phi(\psi)).$$ □

Proposition 9.9. *Assume that $\otimes = *$ is the multiplication or $\otimes = \odot$ is the Łukasiewicz tensor. Then, for every separated representable $\mathcal{U}$-category X, the second inequality of Proposition 9.2 is actually an equality.*

Proof. Let $\Phi : CX \to [0,1]$ in $[0,1]$-FinSup and $\psi \in CX$. Put $u_0 = \bigvee_{x \in X} \psi(x) \otimes \inf_{\psi' \in CX} \hom(\psi'(x), \Phi(\psi'))$ and consider $u_0 < u$. Let $x \in X$. Then

$$u > \psi(x) \otimes \inf_{\psi' \in CX} \hom(\psi'(x), \Phi(\psi')) = \inf_{\psi' \in CX, \psi'(x)=1} \psi(x) \otimes \Phi(\psi'),$$

hence there is some $\psi' \in CX$ with $\psi'(x) = 1$ and

$$\psi(x) \otimes \Phi(\psi') < u.$$

Let now $\alpha < 1$. For every $\psi' \in CX$, we put

$$U_\alpha(\psi') = \{x \in X \mid \psi(x) \otimes \Phi(\psi') < u\} \cap \{x \in X \mid \psi'(x) > \alpha\}.$$

Then $U_\alpha(\psi')$ is open in the compact Hausdorff topology of X, and

$$X = \bigcup_{\psi' \in CX} U_\alpha(\psi').$$

Since X is compact, we find $\psi_1, \dots, \psi_n$ so that

$$X = U_\alpha(\psi_1) \cup \dots \cup U_\alpha(\psi_n).$$

For every $i \in \{1, \ldots, n\}$ we put $D_i = \{x \in X \mid \psi(x) \otimes \Phi(\psi_i) \geq u\}$, then $U_\alpha(\psi_i) \cap D_i = \varnothing$. Let $\widehat{\psi}_i : X \to [0,1]$ be a function (not necessarily a morphism) which is constant 1 on $U_\alpha(\psi_i)$ and constant 0 on D_i. Then, for all $x \in X$,

$$\alpha \otimes \psi(x) \leq (\widehat{\psi}_1(x) \otimes \psi_1(x) \otimes \psi(x)) \vee \cdots \vee (\widehat{\psi}_n(x) \otimes \psi_n(x) \otimes \psi(x)).$$

For every $i \in \{1, \ldots, n\}$ we put $w_i = \sup_{x \in X} \widehat{\psi}_i(x) \otimes \psi(x)$; with the inequality above we get

$$\alpha \otimes \psi \leq (w_1 \otimes \psi_1) \vee \cdots \vee (w_n \otimes \psi_n).$$

Let now $i \in \{1, \ldots, n\}$. Then, for every $x \in X$,

$$\widehat{\psi}_i(x) \otimes \psi(x) \otimes \Phi(\psi_i) \leq u,$$

and therefore $w_i \otimes \Phi(\psi_i) \leq u$. Consequently, $\alpha \otimes \Phi(\psi) \leq u$ for all $\alpha < 1$ and $u > u_0$; which implies $\Phi(\psi) \leq u_0$. □

From Proposition 9.9 we obtain immediately:

Theorem 9.10. *For $\otimes = *$ being the multiplication or $\otimes = \odot$ the Łukasiewicz tensor, the functor*

$$C : \mathfrak{U}\text{-}\mathsf{CogRep}_{\mathbb{V}} \longrightarrow [0,1]\text{-}\mathsf{FinSup}^{\mathrm{op}}$$

is fully faithful.

Our next aim is to identify those morphisms in $[0,1]$-FinSup which correspond to ordinary relations between separated ordered compact spaces on the other side. Recall from Remark 9.1 that, for X being a separated ordered compact Hausdorff space, we can still consider $\psi_1 \otimes \psi_2$ in CX.

Proposition 9.11. *For $\otimes = *$ being the multiplication or $\otimes = \odot$ the Łukasiewicz tensor, a morphism $\varphi : X \mathrel{-\!\!\!\circ\!\!\!\rightarrow} Y$ in $\mathfrak{U}\text{-}\mathsf{Rep}_{\mathbb{V}}$ between separated ordered compact spaces is a 2-distributor if and only if $C\varphi$ satisfies (Ten)$_{\mathrm{lax}}$.*

Proof. Clearly, if $\varphi : X \mathrel{-\!\!\!\circ\!\!\!\rightarrow} Y$ is a 2-distributor, then $C\varphi$ satisfies (Ten)$_{\mathrm{lax}}$. To see the reverse implication, note first that $u \in \{0,1\}$ precisely when $u \leq u \otimes u$. It is enough to consider the case $\varphi : 1 \mathrel{-\!\!\!\circ\!\!\!\rightarrow} X$, and assume now that the corresponding $\Phi : CX \to [0,1]$ satisfies (Ten)$_{\mathrm{lax}}$. Then, for all $x \in X$,

$$\begin{aligned}
\varphi(x) \otimes \varphi(x) &= \Big(\inf_{\psi \in CX, \psi(x)=1} \Phi(\psi)\Big) \otimes \Big(\inf_{\psi' \in CX, \psi'(x)=1} \Phi(\psi')\Big) \\
&= \inf_{\substack{\psi \in CX, \psi(x)=1 \\ \psi' \in CX, \psi'(x)=1}} \Phi(\psi) \otimes \Phi(\psi') \\
&\geq \inf_{\substack{\psi \in CX, \psi(x)=1 \\ \psi' \in CX, \psi'(x)=1}} \Phi(\psi \otimes \psi') \\
&= \inf_{\psi \in CX, \psi(x)=1} \Phi(\psi) = \varphi(x).
\end{aligned}$$

□

The proposition above together with Theorem 9.10 is certainly related to Theorem 6.14; however, in Section 6 we consider finitely cocomplete $\mathcal{V}$-categories equipped with an *additional* monoid structure which is not needed in this section. In fact, Theorem 9.10 allows us to characterise the monoid operation of CX within $[0,1]$-FinSup.

Lemma 9.12. *Assume that $\otimes = *$ is the multiplication or $\otimes = \odot$ is the Łukasiewicz tensor. Let X be a separated ordered compact space and let $\psi_0 \in CX$. Let $\Phi : CX \to CX$ in $[0,1]$-FinSup with $\Phi(1) \leq \psi_0$ and $\Phi(\psi) \leq \psi$, for all $\psi \in CX$. Then $\Phi = \psi_0 \otimes -$ provided that $\psi_0 \otimes \psi \leq \Phi(\psi)$, for all $\psi \in CX$.*

Proof. Let $x \in X$ and consider

$$CX \xrightarrow{\Phi} CX \xrightarrow{\mathrm{ev}_x} [0,1]$$

in $[0,1]$-FinSup. By Theorem 9.10, this arrow corresponds to the continuous $[0,1]$-distributor $\varphi : 1 \mathrel{-\!\!\!\circ\!\!\!\rightarrow} X$ given by

$$\varphi(y) = \inf_{\psi \in CX} \hom(\psi(y), \Phi(\psi)(x)),$$

for all $y \in X$. Let now $y \in X$, we consider the following two cases.

$y \not\geq x$: By Proposition 4.6, there exists some $\psi \in CX$ with $\psi(y) = 1$ and $\psi(x) = 0$. Since $\Phi(\psi) \leq \psi$, we obtain $\Phi(\psi)(x) \leq \psi(x) = 0$; hence $\varphi(y) = 0$.

$y \geq x$: Firstly, for every $\psi \in CX$,

$$\hom(\psi(y), \Phi(\psi)(x)) \geq \hom(\psi(x), \Phi(\psi)(x)) \geq \psi_0(x)$$

since $\psi_0(x) \otimes \psi(x) \leq \Phi(\psi)(x)$. Secondly, $\hom(1, \Phi(1)(x)) \leq \psi_0(x)$. Therefore $\varphi(y) = \psi_0(x)$.

Finally, we obtain

$$\Phi(\psi)(x) = \sup_{y\in X}(\varphi(y) \otimes \psi(y)) = \sup_{y\geq x}(\psi_0(x) \otimes \psi(y)) = \psi_0(x) \otimes \psi(x),$$

for all $\psi \in CX$. $\square$

Theorem 9.13. *Assume that $\otimes = *$ is the multiplication or $\otimes = \odot$ is the Łukasiewicz tensor. Let X be a separated ordered compact space. Then, for every $\psi_0 \in CX$, $\psi_0 \otimes - : CX \to CX$ is the largest morphism $\Phi : CX \to CX$ in $[0,1]$-FinSup satisfying*

$$\Phi(1) \leq \psi_0 \qquad \text{and} \qquad \Phi(\psi) \leq \psi, \text{ for all } \psi \in CX. \tag{9.ii}$$

Proof. Clearly, $\psi_0 \otimes -$ satisfies (9.ii), and the lemma above tells us already that it is maximal among all those maps. Let now $\Phi_1, \Phi_2 : CX \to CX$ be in $[0,1]$-FinSup satisfying (9.ii). Then also the composite arrow

$$CX \xrightarrow{\langle \Phi_1, \Phi_2 \rangle} CX \times CX \xrightarrow{\vee} CX$$

satisfies (9.ii), therefore the collection of all morphism $\Phi : CX \to CX$ in $[0,1]$-FinSup satisfying (9.ii) is directed. Consequently, $\psi_0 \otimes - : CX \to CX$ is the largest such morphism. $\square$

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